

# QUANTUM WALLED BRAUER-CLIFFORD SUPERALGEBRAS

GEORGIA BENKART<sup>1</sup>, NICOLAS GUAY<sup>2</sup>, JI HYE JUNG<sup>3</sup>, SEOK-JIN KANG<sup>4</sup>, AND STEWART WILCOX<sup>5</sup>

**ABSTRACT.** We introduce a new family of superalgebras, the quantum walled Brauer-Clifford superalgebras  $\mathbf{BC}_{r,s}(q)$ . The superalgebra  $\mathbf{BC}_{r,s}(q)$  is a quantum deformation of the walled Brauer-Clifford superalgebra  $\mathbf{BC}_{r,s}$  and a super version of the quantum walled Brauer algebra. We prove that  $\mathbf{BC}_{r,s}(q)$  is the centralizer superalgebra of the action of  $\mathfrak{U}_q(\mathfrak{q}(n))$  on the mixed tensor space  $\mathbf{V}_q^{r,s} = \mathbf{V}_q^{\otimes r} \otimes (\mathbf{V}_q^*)^{\otimes s}$  when  $n \geq r + s$ , where  $\mathbf{V}_q = \mathbb{C}(q)^{(n|n)}$  is the natural representation of the quantum enveloping superalgebra  $\mathfrak{U}_q(\mathfrak{q}(n))$  and  $\mathbf{V}_q^*$  is its dual space. We also provide a diagrammatic realization of  $\mathbf{BC}_{r,s}(q)$  as the  $(r, s)$ -bead tangle algebra  $\mathbf{BT}_{r,s}(q)$ . Finally, we define the notion of  $q$ -Schur superalgebras of type  $\mathbf{Q}$  and establish their basic properties.

## INTRODUCTION

*Schur-Weyl duality* has been one of the most inspiring themes in representation theory, because it reveals many hidden connections between the representation theories of seemingly unrelated algebras. By the duality functor, one algebra appears as the centralizer of the other acting on a common representation space. Many interesting and important algebras have been constructed as centralizer algebras in this way.

For example, the group algebra  $\mathbb{C}\Sigma_k$  of the symmetric group  $\Sigma_k$  appears as the centralizer of the  $\mathfrak{gl}(n)$ -action on  $\mathbf{V}^{\otimes k}$ , where  $\mathbf{V} = \mathbb{C}^n$  is the natural representation of the general linear Lie algebra  $\mathfrak{gl}(n)$ . Similarly, Hecke algebras, Brauer algebras, Birman-Murakami-Wenzl algebras and Hecke-Clifford superalgebras are the centralizer algebras of the action of corresponding Lie (super)algebras or quantum (super)algebras on the tensor powers of their natural representations.

There are further generalizations of Schur-Weyl duality on mixed tensor powers. Let  $\mathbf{V}^{r,s} = \mathbf{V}^{\otimes r} \otimes (\mathbf{V}^*)^{\otimes s}$  be the mixed tensor space of the natural representation  $\mathbf{V}$  of  $\mathfrak{gl}(n)$  and its dual space  $\mathbf{V}^*$ . The centralizer algebra of the  $\mathfrak{gl}(n)$ -action on  $\mathbf{V}^{r,s}$  is the *walled Brauer algebra*  $\mathbf{B}_{r,s}(n)$ . The structure and properties of  $\mathbf{B}_{r,s}(n)$  were first investigated in [1, 12, 20]. By replacing  $\mathfrak{gl}(n)$  by the quantum enveloping algebra  $\mathfrak{U}_q(\mathfrak{gl}(n))$  and  $\mathbf{V} = \mathbb{C}^n$  by  $\mathbf{V}_q = \mathbb{C}(q)^n$ , we obtain as the centralizer algebra the *quantum walled Brauer algebra* studied in [3, 4, 7, 11, 14]. Super versions of the above constructions have been investigated with the following substitutions: Replace  $\mathfrak{gl}(n)$  by  $\mathfrak{gl}(m|n)$ ,  $\mathbb{C}^n$  by  $\mathbb{C}^{(m|n)}$ ;  $\mathfrak{U}_q(\mathfrak{gl}(n))$  by  $\mathfrak{U}_q(\mathfrak{gl}(m|n))$ ; and  $\mathbb{C}(q)^n$  by  $\mathbb{C}(q)^{(m|n)}$  as in [15, 16].

---

2010 *Mathematics Subject Classification.* Primary: 81R50 Secondary: 17B60, 05E10, 57M25, 20G43.

*Key words and phrases.* quantum walled Brauer-Clifford superalgebra, queer Lie superalgebra, centralizer algebra, bead tangle algebra,  $q$ -Schur superalgebra.

<sup>1</sup> G.B. acknowledges with gratitude the hospitality of the Banff International Research Station (BIRS), where her collaboration on this project with N.G. began.

<sup>2</sup> This work was partly supported by a Discovery Grant of the Natural Sciences and Engineering Research Council of Canada.

<sup>3</sup> This work was supported by NRF Grant #2013-035155, NRF-2010-0019516 and NRF-2013R1A1A2063671.

<sup>4</sup> This work was partially supported by NRF Grant #2013-035155 and NRF Grant #2013-055408.

<sup>5</sup> This work was partly supported by a Postdoctoral Fellowship of the Pacific Institute for the Mathematical Sciences.

The Lie superalgebra  $\mathfrak{q}(n)$  is commonly referred to as the *queer Lie superalgebra* because of its unique properties and the fact that it has no non-super counterpart. Its natural representation is the superspace ( $\mathbb{Z}_2$ -graded vector space)  $\mathbf{V} = \mathbb{C}^{(n|n)}$ . The corresponding centralizer algebra  $\text{End}_{\mathfrak{q}(n)}(\mathbf{V}^{\otimes r})$  was studied by Sergeev in [19], and it is often referred to as the *Sergeev algebra*. Using a modified version of a technique of Fadeev, Reshetikhin and Turaev, Olshanski introduced the *quantum queer superalgebra*  $\mathfrak{U}_q(\mathfrak{q}(n))$  and established an analogue of Schur-Weyl duality. That is, he showed that there is a surjective algebra homomorphism  $\rho_{n,q}^r : \text{HC}_r(q) \rightarrow \text{End}_{\mathfrak{q}(n)}(\mathbf{V}^{\otimes r})$ , where  $\text{HC}_r(q)$  is the *Hecke-Clifford superalgebra*, a quantum version of the Sergeev algebra. Moreover,  $\rho_{n,q}^r$  is an isomorphism when  $n \geq r$ .

On the other hand, in [13] Jung and Kang considered a super version of the walled Brauer algebra. For the mixed tensor space  $\mathbf{V}^{r,s} = \mathbf{V}^{\otimes r} \otimes (\mathbf{V}^*)^{\otimes s}$ , one can ask, What is the centralizer of the  $\mathfrak{q}(n)$ -action on  $\mathbf{V}^{r,s}$ ? In order to answer this question, they introduced two versions of the *walled Brauer-Clifford superalgebra*, (which is called the *walled Brauer superalgebra* in [13]). The first is constructed using  $(r, s)$ -superdiagrams, and the second is defined by generators and relations. The main results of [13] show that these two definitions are equivalent and that there is a surjective algebra homomorphism  $\rho_n^{r,s} : \text{BC}_{r,s} \rightarrow \text{End}_{\mathfrak{q}(n)}(\mathbf{V}^{r,s})$ , which is an isomorphism whenever  $n \geq r + s$ .

The purpose of this paper is to combine the constructions in [18] and [13] to determine the centralizer algebra of the  $\mathfrak{U}_q(\mathfrak{q}(n))$ -action on the mixed tensor space  $\mathbf{V}_q^{r,s} := \mathbf{V}_q^{\otimes r} \otimes (\mathbf{V}_q^*)^{\otimes s}$ . We begin by introducing the *quantum walled Brauer-Clifford superalgebra*  $\text{BC}_{r,s}(q)$  via generators and relations. The superalgebra  $\text{BC}_{r,s}(q)$  contains the quantum walled Brauer algebra and the Hecke-Clifford superalgebra  $\text{HC}_r(q)$  as subalgebras.

We then define an action of  $\text{BC}_{r,s}(q)$  on the mixed tensor space  $\mathbf{V}_q^{r,s}$  that supercommutes with the action of  $\mathfrak{U}_q(\mathfrak{q}(n))$ . As a result, there is a superalgebra homomorphism  $\rho_{n,q}^{r,s} : \text{BC}_{r,s}(q) \rightarrow \text{End}_{\mathfrak{U}_q(\mathfrak{q}(n))}(\mathbf{V}_q^{r,s})$ . Actually, defining such an action is quite subtle and complicated, and we regard this as one of our main results (Theorem 3.16). We use the fact that  $\text{BC}_{r,s}$  is the classical limit of  $\text{BC}_{r,s}(q)$  to show that the homomorphism  $\rho_{n,q}^{r,s}$  is surjective and that it is an isomorphism whenever  $n \geq r + s$  (Theorem 3.28).

We also give a diagrammatic realization of  $\text{BC}_{r,s}(q)$  as the  $(r, s)$ -bead tangle algebra  $\text{BT}_{r,s}(q)$ . An  $(r, s)$ -bead tangle is a portion of a planar knot diagram satisfying the conditions in Definition 4.1. The algebra  $\text{BT}_{r,s}(q)$  is a quantum deformation of the  $(r, s)$ -bead diagram algebra  $\text{BD}_{r,s}$ , which is isomorphic to the walled Brauer-Clifford superalgebra  $\text{BC}_{r,s}$  (Theorem 2.9). Modifying the arguments in [10], we prove that the algebra  $\text{BT}_{r,s}(q)$  is isomorphic to  $\text{BC}_{r,s}(q)$  (Theorem 4.19), so that  $\text{BC}_{r,s}(q)$  can be regarded as a diagram algebra.

In the final section, we introduce *q-Schur superalgebras of type Q* and prove that the classical results for *q-Schur algebras* can be extended to this setting.

## 1. THE WALLED BRAUER-CLIFFORD SUPERALGEBRAS

To begin, we recall the definition of the Lie superalgebra  $\mathfrak{q}(n)$  and its basic properties. Let  $\mathbf{I} = \{\pm i \mid i = 1, \dots, n\}$ , and set  $\mathbb{Z}_2 = \mathbb{Z}/2\mathbb{Z}$ . The superspace  $\mathbf{V} = \mathbb{C}(n|n) = \mathbb{C}^n \oplus \mathbb{C}^n$  has a standard basis  $\{v_i \mid i \in \mathbf{I}\}$ . We say that the *parity* of  $v_i$  equals  $|i| := |v_i|$ , where  $|v_i| = 1$  if  $i < 0$  and  $|v_i| = 0$  if  $i > 0$ .

The endomorphism algebra is  $\mathbb{Z}_2$ -graded,  $\text{End}_{\mathbb{C}}(\mathbf{V}) = \text{End}_{\mathbb{C}}(\mathbf{V})_0 \oplus \text{End}_{\mathbb{C}}(\mathbf{V})_1$ , and has a basis of matrix units  $E_{ij}$  with  $-n \leq i, j \leq n$ ,  $ij \neq 0$ , where the parity of  $E_{ij}$  is  $|E_{ij}| = |i| + |j| \pmod{2}$ . The general linear Lie superalgebra  $\mathfrak{gl}(n|n)$  is obtained from  $\text{End}_{\mathbb{C}}(\mathbf{V})$  by using the *supercommutator*

$$[X, Y] = XY - (-1)^{|X||Y|} YX$$

for homogeneous elements  $X, Y$ . The map  $\iota : \mathfrak{gl}(n|n) \rightarrow \mathfrak{gl}(n|n)$  given by  $E_{ij} \mapsto E_{-i, -j}$  is an involutive automorphism of  $\mathfrak{gl}(n|n)$ . Let  $J = \sum_{a=1}^n (E_{a, -a} - E_{-a, a}) \in \mathfrak{gl}(n|n)$ .

**Definition 1.1.** The *queer Lie superalgebra*  $\mathfrak{q}(n)$  can be defined equivalently as either the centralizer of  $J$  in  $\mathfrak{gl}(n|n)$  (under the supercommutator product) or the fixed-point subalgebra of  $\mathfrak{gl}(n|n)$  with respect to the involution  $\iota$ .

Identifying  $\mathbf{V}$  with the space of  $(n|n)$  column vectors and  $\{v_i \mid i \in \mathbf{I}\}$  with the standard basis for the column vectors, we have  $J = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}$ , and  $\mathfrak{q}(n)$  can be expressed in the matrix form as

$$\left\{ \begin{pmatrix} A & B \\ B & A \end{pmatrix} \mid A, B \text{ are arbitrary } n \times n \text{ complex matrices} \right\}.$$

Then  $\mathfrak{q}(n)$  inherits a  $\mathbb{Z}_2$ -grading from  $\mathfrak{gl}(n|n)$ , and a basis for  $\mathfrak{q}(n)_0$  is given by  $E_{ab}^0 = E_{ab} + E_{-a, -b}$  and for  $\mathfrak{q}(n)_1$  by  $E_{ab}^1 = E_{a, -b} + E_{-a, b}$ , where  $1 \leq a, b \leq n$ .

The superalgebra  $\mathfrak{q}(n)$  acts naturally on  $\mathbf{V}$  by matrix multiplication on the column vectors, and  $\mathbf{V}$  is an irreducible representation of  $\mathfrak{q}(n)$ . The action on  $\mathbf{V}$  extends to one on  $\mathbf{V}^{\otimes k}$  by

$$(1.2) \quad g(v_{i_1} \otimes \cdots \otimes v_{i_k}) = \sum_{j=1}^k (-1)^{(|v_{i_1}| + \cdots + |v_{i_{j-1}}|)|g|} v_{i_1} \otimes \cdots \otimes v_{i_{j-1}} \otimes g v_{i_j} \otimes v_{i_{j+1}} \otimes \cdots \otimes v_{i_k},$$

where  $g$  is homogeneous. It also extends in a similar fashion to the mixed tensor space  $\mathbf{V}^{r,s} := \mathbf{V}^{\otimes r} \otimes (\mathbf{V}^*)^{\otimes s}$ , where  $\mathbf{V}^*$  is the dual representation of  $\mathbf{V}$ , and the action on  $\mathbf{V}^*$  is given by

$$(g\omega)(v) := -(-1)^{|g||\omega|} \omega(gv)$$

for homogeneous elements  $g \in \mathfrak{q}(n)$ ,  $\omega \in \mathbf{V}^*$ , and  $v \in \mathbf{V}$ . We assume  $\{\omega_i \mid i \in \mathbf{I}\}$  is the basis of  $\mathbf{V}^*$  dual to the standard basis  $\{v_i \mid i \in \mathbf{I}\}$  of  $\mathbf{V}$ .

In an effort to construct the centralizer superalgebra  $\text{End}_{\mathfrak{q}(n)}(\mathbf{V}^{r,s})$ , Jung and Kang [13] introduced the notion of the *walled Brauer-Clifford superalgebra*  $\text{BC}_{r,s}$ . The superalgebra  $\text{BC}_{r,s}$  is generated by even generators  $s_1, \dots, s_{r-1}, s_{r+1}, \dots, s_{r+s-1}, e_{r,r+1}$  and odd generators  $c_1, \dots, c_{r+s}$ , which satisfy the following defining relations (for  $i, j$  in the allowable range):

$$(1.3) \quad \begin{aligned} s_i^2 &= 1, \quad s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1}, \quad s_i s_j = s_j s_i \quad (|i - j| > 1), \\ e_{r,r+1}^2 &= 0, \quad e_{r,r+1} s_j = s_j e_{r,r+1} \quad (j \neq r-1, r+1), \\ e_{r,r+1} &= e_{r,r+1} s_{r-1} e_{r,r+1} = e_{r,r+1} s_{r+1} e_{r,r+1}, \\ s_{r-1} s_{r+1} e_{r,r+1} s_{r+1} s_{r-1} e_{r,r+1} &= e_{r,r+1} s_{r-1} s_{r+1} e_{r,r+1} s_{r+1} s_{r-1}, \\ c_i^2 &= -1 \quad (1 \leq i \leq r), \quad c_i^2 = 1 \quad (r+1 \leq i \leq r+s), \quad c_i c_j = -c_j c_i \quad (i \neq j), \\ s_i c_i s_i &= c_{i+1}, \quad s_i c_j = c_j s_i \quad (j \neq i, i+1), \\ c_r e_{r,r+1} &= c_{r+1} e_{r,r+1}, \quad e_{r,r+1} c_r = e_{r,r+1} c_{r+1}, \\ e_{r,r+1} c_r e_{r,r+1} &= 0, \quad e_{r,r+1} c_j = c_j e_{r,r+1} \quad (j \neq r, r+1). \end{aligned}$$

For  $1 \leq j \leq r-1$ ,  $r+1 \leq k \leq r+s-1$ ,  $1 \leq l \leq r$ , and  $r+1 \leq m \leq r+s$ , the action of the generators of  $\text{BC}_{r,s}$  on  $\mathbf{V}^{r,s}$  (which is on the right) is defined as follows:

$$\begin{aligned} (v_{i_1} \otimes \cdots \otimes v_{i_r} \otimes \omega_{i_{r+1}} \otimes \cdots \otimes \omega_{i_{r+s}}) s_j &= (-1)^{|i_j||i_{j+1}|} v_{i_1} \otimes \cdots \otimes v_{i_{j-1}} \otimes v_{i_{j+1}} \otimes v_{i_j} \otimes v_{i_{j+2}} \cdots \otimes \omega_{i_{r+s}}, \\ (v_{i_1} \otimes \cdots \otimes v_{i_r} \otimes \omega_{i_{r+1}} \otimes \cdots \otimes \omega_{i_{r+s}}) s_k &= (-1)^{|i_k||i_{k+1}|} v_{i_1} \otimes \cdots \otimes \omega_{i_{k-1}} \otimes \omega_{i_{k+1}} \otimes \omega_{i_k} \otimes \omega_{i_{k+2}} \cdots \otimes \omega_{i_{r+s}}, \\ (v_{i_1} \otimes \cdots \otimes v_{i_r} \otimes \omega_{i_{r+1}} \otimes \cdots \otimes \omega_{i_{r+s}}) e_{r,r+1} &= \delta_{i_r, i_{r+1}} (-1)^{|i_r|} \sum_{i=-n}^n v_{i_1} \otimes \cdots \otimes v_{i_{r-1}} \otimes v_i \otimes \omega_i \otimes \omega_{i_{r+2}} \cdots \otimes \omega_{i_{r+s}}, \end{aligned}$$

$(v_{i_1} \otimes \cdots \otimes v_{i_r} \otimes \omega_{i_{r+1}} \otimes \cdots \otimes \omega_{i_{r+s}}) c_l = (-1)^{|i_1| + \cdots + |i_{l-1}|} v_{i_1} \otimes \cdots \otimes v_{i_{l-1}} \otimes J v_{i_l} \otimes v_{i_{l+1}} \otimes \cdots \otimes \omega_{i_{r+s}},$   
 $(v_{i_1} \otimes \cdots \otimes v_{i_r} \otimes \omega_{i_{r+1}} \otimes \cdots \otimes \omega_{i_{r+s}}) c_m = (-1)^{|i_1| + \cdots + |i_{m-1}|} v_{i_1} \otimes \cdots \otimes \omega_{i_{m-1}} \otimes J^T \omega_{i_m} \otimes \omega_{i_{m+1}} \otimes \cdots \otimes \omega_{i_{r+s}},$   
 where  $J^T$  is the supertranspose of  $J$ , and the *supertranspose* is given by  $E_{ij}^T := (-1)^{(|i|+|j|)|i|} E_{ji}$ .  
 By direct calculation, it can be verified that this action of the generators gives rise to an action of  $\text{BC}_{r,s}$  on  $\mathbf{V}^{r,s}$ . Thus, there is an homomorphism of superalgebras,

$$\rho_n^{r,s} : \text{BC}_{r,s} \longrightarrow \text{End}_{\mathbb{C}}(\mathbf{V}^{r,s})^{\text{op}}.$$

The next theorem, due to Jung and Kang [13], identifies a basis for  $\text{BC}_{r,s}$ . In stating it and some subsequent results, we use the following notation: For a nonempty subset  $A = \{a_1 < \cdots < a_m\}$  of  $\{1, \dots, r+s\}$ , let  $c_A = c_{a_1} \cdots c_{a_m}$ , and set  $c_\emptyset = 1$ . Let  $e_{p,q} = \varphi e_{r,r+1} \varphi^{-1}$ , where  $\varphi = s_{q-1} \cdots s_{r+1} s_p \cdots s_{r-1}$  for  $1 \leq p \leq r$  and  $r+1 \leq q \leq r+s$ .

**Theorem 1.4.**

- (i) The actions of  $\text{BC}_{r,s}$  and  $\mathfrak{q}(n)$  on  $\mathbf{V}^{r,s}$  supercommute with each other. Thus, there is an algebra homomorphism  $\rho_n^{r,s} : \text{BC}_{r,s} \rightarrow \text{End}_{\mathfrak{q}(n)}(\mathbf{V}^{r,s})^{\text{op}}$ .
- (ii) The map  $\rho_n^{r,s}$  is surjective. Moreover, it is an isomorphism if  $n \geq r+s$ .
- (iii) The elements

$$c_P e_{p_1, q_1} \cdots e_{p_a, q_a} \sigma c_Q$$

such that

- (1)  $1 \leq p_1 < \cdots < p_a \leq r$ ;
  - (2)  $r+1 \leq q_i \leq r+s$ ,  $i = 1, \dots, a$ , are all distinct;
  - (3)  $\sigma \in \sum_r \times \sum_s$ ,  $\sigma^{-1}(p_1) < \cdots < \sigma^{-1}(p_a)$ ; and
  - (4)  $P \subseteq \{p_1, \dots, p_a\}$ ,  $Q \subseteq \{1, \dots, r\} \cup \{r+1, \dots, r+s\} \setminus \{\sigma^{-1}(q_1), \dots, \sigma^{-1}(q_a)\}$
- comprise a basis of  $\text{BC}_{r,s}$ .
- (iv)  $\dim_{\mathbb{C}} \text{BC}_{r,s} = 2^{r+s} (r+s)!$ .

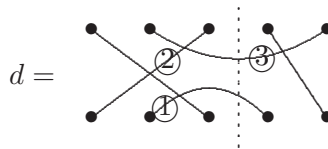
## 2. THE $(r, s)$ -BEAD DIAGRAM ALGEBRA $\text{BD}_{r,s}$

In this section, we give a new diagrammatic realization of the walled Brauer-Clifford superalgebra  $\text{BC}_{r,s}$ .

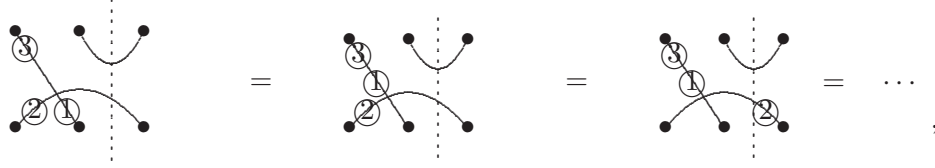
**Definition 2.1.** An  $(r, s)$ -bead diagram, or simply a *bead diagram*, is a graph consisting of two rows with  $r+s$  vertices in each row such that the following conditions hold:

- (1) Each vertex is connected by a strand to exactly one other vertex.
- (2) Each strand may (or may not) have finitely many numbered beads. The bead numbers on a diagram start with 1 and are distinct consecutive positive integers.
- (3) There is a vertical wall separating the first  $r$  vertices from the last  $s$  vertices in each row.
- (4) A *vertical strand* connects a vertex on the top row with one on the bottom row, and it cannot cross the wall. A *horizontal strand* connects vertices in the same row, and it must cross the wall.
- (5) No loops are permitted in an  $(r, s)$ -bead diagram.

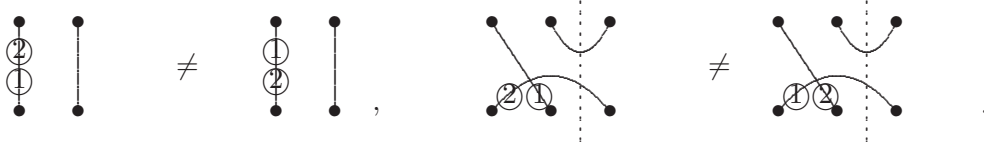
The following diagram is an example of a  $(3, 2)$ -bead diagram.



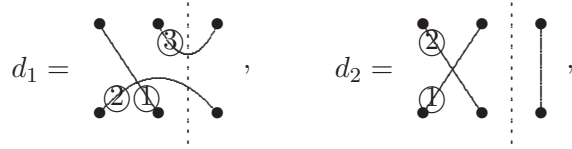
Beads can slide along a given strand, but they cannot jump to another strand nor can they interchange positions on a given strand. For example, the following diagrams are the same as  $(2, 1)$ -bead diagrams.



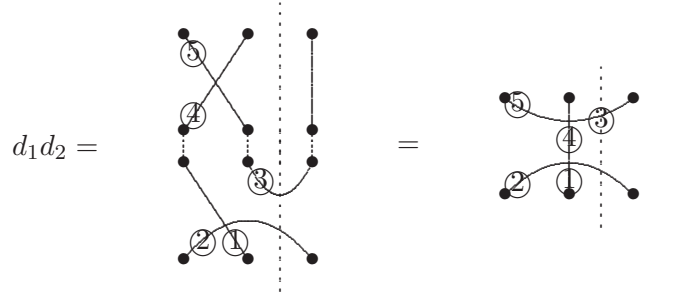
while the following diagrams are considered to be different



An  $(r, s)$ -bead diagram having an even number of beads is regarded as *even* (resp. *odd*). Let  $\widetilde{\text{BD}}_{r,s}$  be the superspace with basis consisting of the  $(r, s)$ -bead diagrams. We define a multiplication on  $\widetilde{\text{BD}}_{r,s}$ . For  $(r, s)$ -bead diagrams  $d_1, d_2$ , we put  $d_1$  under  $d_2$  and identify the top vertices of  $d_1$  with the bottom vertices of  $d_2$ . If there is a loop in the middle row, we say  $d_1 d_2 = 0$ . If there is no loop in the middle row, we add the largest bead number in  $d_1$  to each bead number in  $d_2$ , so that if  $m$  is the largest bead number in  $d_1$ , then a bead numbered  $i$  in  $d_2$  is now numbered  $m + i$  in  $d_1 d_2$ . Then we concatenate the diagrams. For example, if



then



Observe that  $\widetilde{\text{BD}}_{r,s}$  is closed under this product. If the number of beads in  $d_1$  (resp.  $d_2$ ) is  $m_1$  (resp.  $m_2$ ) and  $d_1 d_2 \neq 0$ , then the number of beads in  $d_1 d_2$  is  $m_1 + m_2$ . Hence, the multiplication respects the  $\mathbb{Z}_2$ -grading. Let  $d_1, d_2, d_3 \in \widetilde{\text{BD}}_{r,s}$ . Note that the connections in  $(d_1 d_2) d_3$  and  $d_1 (d_2 d_3)$  are the same. The strands where the beads are placed and the bead numbers are also the same in  $(d_1 d_2) d_3$  and  $d_1 (d_2 d_3)$ . Therefore,  $(d_1 d_2) d_3 = d_1 (d_2 d_3)$ . The identity element of  $\widetilde{\text{BD}}_{r,s}$  is the diagram with no beads such that each top vertex is connected to the corresponding bottom vertex.

For  $1 \leq i \leq r - 1$ ,  $r + 1 \leq j \leq r + s - 1$ ,  $1 \leq k \leq r$ ,  $r + 1 \leq l \leq r + s$ , let  $\text{BD}'_{r,s}$  be the subalgebra of  $\widetilde{\text{BD}}_{r,s}$  generated by the following diagrams:

$$\begin{aligned}
\mathbf{s}_i &:= \begin{array}{c} \text{Diagram with strands } 1, \dots, i-1, i, i+1, \dots, r+s. \text{ Strands } i \text{ and } i+1 \text{ cross.} \end{array}, \quad \mathbf{s}_j := \begin{array}{c} \text{Diagram with strands } 1, \dots, j-1, j, j+1, \dots, r+s. \text{ Strands } j \text{ and } j+1 \text{ cross.} \end{array}, \\
\mathbf{e}_{r,r+1} &:= \begin{array}{c} \text{Diagram with strands } 1, \dots, r, r+1, \dots, r+s. \text{ Strands } r \text{ and } r+1 \text{ are connected by two arcs.} \end{array}, \\
\mathbf{c}_k &:= \begin{array}{c} \text{Diagram with strands } 1, \dots, k, \dots, r+s. \text{ Strand } k \text{ has a bead labeled 1.} \end{array}, \quad \mathbf{c}_l := \begin{array}{c} \text{Diagram with strands } 1, \dots, l, \dots, r+s. \text{ Strand } l \text{ has a bead labeled 1.} \end{array}.
\end{aligned}$$

Assume  $L$  is the (two-sided) ideal of  $\text{BD}'_{r,s}$  generated by the following homogeneous elements,

$$(2.2) \quad \mathbf{c}_k^2 + 1 \quad (1 \leq k \leq r), \quad \mathbf{c}_l^2 - 1 \quad (r+1 \leq l \leq r+s), \quad \text{and} \quad \mathbf{c}_i \mathbf{c}_j + \mathbf{c}_j \mathbf{c}_i \quad (1 \leq i \neq j \leq r+s),$$

and let  $\text{BD}_{r,s}$  be the quotient superalgebra  $\text{BD}'_{r,s}/L$ . We say that  $\text{BD}_{r,s}$  is the  $(r,s)$ -bead diagram algebra, or simply the *bead diagram algebra*. For simplicity, we identify cosets in  $\text{BD}_{r,s}$  with their diagram representatives and make the following identifications in  $\text{BD}_{r,s}$ :

$$\begin{aligned}
\mathbf{c}_k^2 &= \begin{array}{c} \text{Diagram with strands } 1, \dots, k, \dots, r+s. \text{ Strand } k \text{ has two beads labeled 1 and 2.} \end{array} = - \begin{array}{c} \text{Diagram with strands } 1, \dots, k, \dots, r+s. \end{array}, \\
\mathbf{c}_l^2 &= \begin{array}{c} \text{Diagram with strands } 1, \dots, l, \dots, r+s. \text{ Strand } l \text{ has two beads labeled 1 and 2.} \end{array} = \begin{array}{c} \text{Diagram with strands } 1, \dots, l, \dots, r+s. \end{array},
\end{aligned}$$

and

$$\mathbf{c}_i \mathbf{c}_j = \begin{array}{c} \text{Diagram with strands } 1, \dots, i, \dots, j, \dots, r+s. \text{ Strand } i \text{ has a bead labeled 1, strand } j \text{ has a bead labeled 2.} \end{array} = - \begin{array}{c} \text{Diagram with strands } 1, \dots, i, \dots, j, \dots, r+s. \text{ Strand } i \text{ has a bead labeled 2, strand } j \text{ has a bead labeled 1.} \end{array} = -\mathbf{c}_j \mathbf{c}_i.$$

Our aim is to prove that the walled Brauer-Clifford superalgebra  $\text{BC}_{r,s}$  is isomorphic to  $\text{BD}_{r,s}$ . Towards this purpose, we adopt the following conventions:

- (i) The vertices on the top row (and on the bottom row) of a bead diagram are labeled  $1, 2, \dots, r+s$  from left to right.
- (ii) The top vertex of a vertical strand or the left vertex of a horizontal strand is the *good* vertex of the strand.
- (iii) A bead on a horizontal bottom row strand is a bead of *type I*. All other beads are of *type II*.

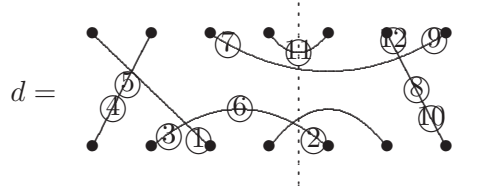
We construct a bead diagram  $\tilde{d}$  from the bead diagram  $d$  by performing the following steps:

- (1) Keep the same connections between vertices as in  $d$ .

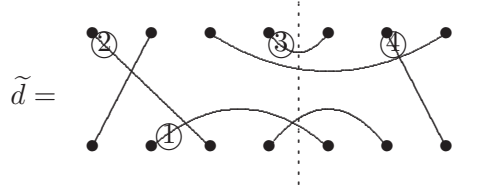
- (2) If the number of beads along a strand is even, delete all beads on that strand. If there is an odd number of beads on a strand, leave only one bead on it. Repeat this process for all strands in  $d$ . (Hence, there is at most one bead on any strand of  $\tilde{d}$ .) Associate to each remaining bead the good vertex of its strand.
- (3) Renumber (starting with 1) the beads of type I according to the position of its good vertex from left to right.
- (4) Let  $m$  be the maximum of bead numbers after Step 3. Renumber (starting with  $m + 1$ ) the beads of type II according to the position of its good vertex from left to right.

The resulting diagram is  $\tilde{d}$ . Since the definition of  $\tilde{d}$  depends only on the number of beads along a strand and the good vertices of strands having an odd number of beads,  $\tilde{d}$  does not change when we slide beads along a strand, so  $\tilde{d}$  is well defined.

**Example 2.3.** If  $d$  is as pictured below



then



Next we assign a nonnegative integer  $\gamma(d)$  to the bead diagram  $d$  in the following way:

- (1) Assume the bead numbers of type I in  $d$  are  $1 \leq \eta_1 < \eta_2 < \dots < \eta_p$ . Let  $a_j$  be the label of the good vertex of the strand in  $d$  with the bead  $\eta_j$ . This determines a sequence  $a_1, \dots, a_p$ . Let  $\ell_1(d) := |\{(j, k) \mid j < k, a_j > a_k\}|$ .
- (2) Assume the bead numbers of type II in  $d$  are  $1 \leq \vartheta_1 < \vartheta_2 < \dots < \vartheta_q$ . Let  $b_j$  be the label of the good vertex of the strand in  $d$  with the bead  $\vartheta_j$ . This determines a sequence  $b_1, \dots, b_q$ . Let  $\ell_2(d) := |\{(j, k) \mid j < k, b_j > b_k\}|$ .
- (3) Let  $\rho_1(d) := \sum_{i=1}^r \left\lfloor \frac{|\{j \in \{1, \dots, p\} \mid a_j = i\}|}{2} \right\rfloor$ , where  $\lfloor x \rfloor$  denotes the largest integer not greater than  $x$ .
- (4) Let  $\rho_2(d) = \sum_{i=1}^r \left\lfloor \frac{|\{j \in \{1, \dots, q\} \mid b_j = i\}|}{2} \right\rfloor$ .
- (5) For each bead  $\eta_j$ , its *passing number* counts the number of beads  $\eta_k$  such that  $\eta_k > \eta_j$  when  $\eta_j$  moves to the good vertex of its strand. Let  $p_1(d)$  be the sum of the passing numbers for all  $\eta_1, \dots, \eta_p$ .
- (6) For each bead  $\vartheta_j$ , its *passing number* counts the number of beads  $\vartheta_k$  with  $\vartheta_k < \vartheta_j$  when  $\vartheta_j$  moves to the good vertex of its strand. Let  $p_2(d)$  be the sum of the passing numbers for all  $\vartheta_1, \dots, \vartheta_q$ .
- (7) For each bead  $\vartheta_j$ , count the number of beads  $\eta_k$  such that  $\eta_k > \vartheta_j$ ; then let  $c(d)$  be the sum of those numbers for all beads  $\vartheta_1, \dots, \vartheta_q$ .



$$(8) \text{ Let } \alpha(d) := \sum_{i=r+1}^{r+s} \left\lfloor \frac{|\{j \in \{1, \dots, q\} \mid b_j = i\}|}{2} \right\rfloor.$$

$$(9) \text{ Now set } \beta(d) := \ell_1(d) + \ell_2(d) + \rho_1(d) + \rho_2(d) + p_1(d) + p_2(d) + c(d) \text{ and } \gamma(d) := \beta(d) + \alpha(d).$$

Since the definition of  $\gamma(d)$  depends only on the bead numbers, the number of beads on a strand, and the good vertex of a strand having a bead,  $\gamma(d)$  is well defined. All values including  $\beta(d)$ ,  $\alpha(d)$ , and hence  $\gamma(d)$ , are nonnegative integers.

**Example 2.4.** Consider the bead diagram  $d$  in Example 2.3. Three beads ②, ③, ⑥ are of type I, and the rest are of type II. The sequence of labels for the good vertices obtained in Step 1 (resp. Step 2) is  $a_1, a_2, a_3 = 2, 2, 2$  (resp.  $b_1, \dots, b_9 = 1, 2, 2, 3, 6, 3, 6, 4, 6$ ). From these sequences we see that

$$\ell_1(d) = 0, \quad \ell_2(d) = 3, \quad \rho_1(d) = 1, \quad \rho_2(d) = 2, \quad \text{and } \alpha(d) = 1.$$

When the bead ② moves to the good vertex on its strand, it must pass the two beads ③, ⑥. When the beads ③ and ⑥ move to that same good vertex, they do not have to pass a bead with a larger label. Hence  $p_1(d) = 2$ . Similarly  $p_2(d) = 2$ , since only the beads ⑨ (passing ⑦) and ⑩ (passing ⑧) contribute to  $p_2(d)$ .

Only the following beads contribute to  $c(d)$ : bead ① with ②, ③, ⑥ and beads ④ and ⑤ with ⑥. Therefore,  $c(d) = 5$ .

Consequently,  $\beta(d) = 3 + 1 + 2 + 2 + 2 + 5 = 15$ ,  $\alpha(d) = 1$ , and  $\gamma(d) = 16$ .

The set of  $(r, s)$ -bead diagrams without any beads or horizontal strands forms a group under the multiplication defined in  $\widetilde{\text{BD}}_{r,s}$  which is isomorphic to the product  $\Sigma_r \times \Sigma_s$  of symmetric groups. In what follows, we identify that group with  $\Sigma_r \times \Sigma_s$  but use boldface when we are regarding an element of  $\Sigma_r \times \Sigma_s$  as a diagram. We adopt the following conventions analogous to those for the basis elements of  $\text{BC}_{r,s}$ , but here we are using the generators for the subalgebra  $\text{BD}'_{r,s}$  of  $\widetilde{\text{BD}}_{r,s}$ :

For a nonempty subset  $A = \{a_1 < \dots < a_m\}$  of  $\{1, \dots, r\} \cup \{r+1, \dots, r+s\}$ , set  $\mathbf{c}_A := \mathbf{c}_{a_1} \cdots \mathbf{c}_{a_m}$ , and let  $\mathbf{c}_\emptyset = 1$ . Let  $\mathbf{e}_{p,q} := \boldsymbol{\varphi} \mathbf{e}_{r,r+1} \boldsymbol{\varphi}^{-1}$ , where  $\boldsymbol{\varphi} = \mathbf{s}_{q-1} \cdots \mathbf{s}_{r+1} \mathbf{s}_p \cdots \mathbf{s}_{r-1}$  for  $1 \leq p \leq r$  and  $r+1 \leq q \leq r+s$ .

**Lemma 2.5.** (i) For a bead diagram  $d$ , the associated diagram  $\tilde{d}$  has an expression of the form

$$\mathbf{c}_P \mathbf{e}_{p_1, q_1} \cdots \mathbf{e}_{p_a, q_a} \boldsymbol{\sigma} \mathbf{c}_Q,$$

where

- (1)  $1 \leq p_1 < \dots < p_a \leq r$ ;
- (2)  $r+1 \leq q_i \leq r+s$ ,  $i = 1, \dots, a$ , are all distinct;
- (3)  $\boldsymbol{\sigma} \in \Sigma_r \times \Sigma_s$ ,  $\sigma^{-1}(p_1) < \dots < \sigma^{-1}(p_a)$ ; and
- (4)  $P \subseteq \{p_1, \dots, p_a\}$ ,  $Q \subseteq \{1, \dots, r\} \cup \{r+1, \dots, r+s\} \setminus \{\sigma^{-1}(q_1), \dots, \sigma^{-1}(q_a)\}$ .

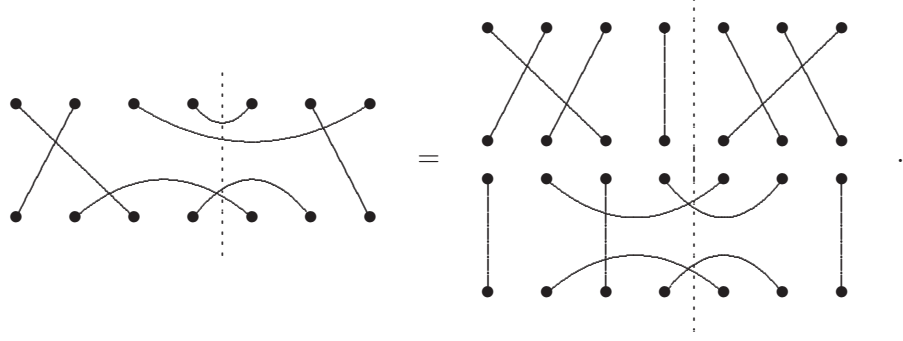
Hence,  $\tilde{d} \in \text{BD}'_{r,s}$ .

(ii)  $\gamma(d) = 0$  if and only if  $d = \tilde{d}$ . In particular,  $\gamma(\tilde{d}) = 0$  for all bead diagrams  $d$ .

*Proof.* (i) A diagram without beads can be written as a product  $\mathbf{e}_{p_1, q_1} \cdots \mathbf{e}_{p_a, q_a} \boldsymbol{\sigma}$  which satisfies conditions (1), (2), and (3). Indeed,  $a$  is the number of horizontal strands in  $\tilde{d}$ . The number  $p_i$  (resp.  $q_i$ ) is the label of the good vertex (resp. right vertex) of the  $i$ th horizontal bottom row strand from left to right. The number  $\sigma^{-1}(p_i)$  (resp.  $\sigma^{-1}(q_i)$ ) is the label of the good vertex (resp. right



vertex) of the  $i$ th horizontal top row strand from left to right. For example,



From Steps 3 and 4 of the construction of  $\tilde{d}$  from  $d$ , we have  $\tilde{d} = \mathbf{c}_P \mathbf{e}_{p_1, q_1} \cdots \mathbf{e}_{p_a, q_a} \sigma \mathbf{c}_Q$ , where  $P$  is a set of labels for the good vertices of strands with beads of type I, and  $Q$  is a set of labels for the good vertices of strands with beads of type II in  $\tilde{d}$ . Since we can slide a bead to the good vertex on its strand, condition (4) above can be satisfied.

(ii) ( $\Rightarrow$ ) From the assumption that  $\gamma(d) = 0$ , we have that  $\rho_1(d) = \rho_2(d) = \alpha(d) = 0$ . Hence, there is at most one bead on each strand in  $d$ . Since  $c(d) = 0$ , the bead numbers of the beads of type I are less than all the bead numbers of the beads of type II. From  $\ell_1(d) = \ell_2(d) = 0$ , we deduce that the sequence of good vertices for the stands with beads of the same type are arranged in increasing size. That is, the good vertex having the smaller label is connected to the strand with the bead having the smaller bead number. From these properties, we determine that nothing is changed when  $\tilde{d}$  is constructed from  $d$ , so that  $\tilde{d} = d$ .

( $\Leftarrow$ ) It is enough to argue that  $\gamma(\tilde{d}) = 0$ . By Step 2 of the construction of  $\tilde{d}$ , we have  $\rho_1(\tilde{d}) = \rho_2(\tilde{d}) = \alpha(\tilde{d}) = 0$ . Also, since there is at most one bead on each strand in  $\tilde{d}$ ,  $p_1(\tilde{d}) = p_2(\tilde{d}) = 0$ . By Steps 3 and 4, we see that  $c(\tilde{d}) = \ell_1(\tilde{d}) = \ell_2(\tilde{d}) = 0$ . As a result,  $\beta(\tilde{d}) = \alpha(\tilde{d}) = 0$ , and hence  $\gamma(\tilde{d}) = 0$ .  $\square$

**Example 2.6.** For the diagram  $\tilde{d}$  in Example 2.3, we have  $p_1 = 2, p_2 = 4, q_1 = 5, q_2 = 6$ ,  $\sigma = s_2 s_1 s_5 s_6$ ,  $\sigma^{-1}(2) = 3, \sigma^{-1}(4) = 4, \sigma^{-1}(5) = 7, \sigma^{-1}(6) = 5$ ,  $P = \{2\}$ , and  $Q = \{1, 4, 6\}$ . Consequently,  $\tilde{d} = \mathbf{c}_2 \mathbf{e}_{2,5} \mathbf{e}_{4,6} s_2 s_1 s_5 s_6 \mathbf{c}_1 \mathbf{c}_4 \mathbf{c}_6 \in \text{BD}'_{4,3}$ .

**Lemma 2.7.** The subspace  $\mathbf{M}$  spanned by  $\{d - (-1)^{\beta(d)} \tilde{d} \mid d \text{ is a bead diagram in } \text{BD}'_{r,s}\}$  contains the two-sided ideal  $\mathbf{L}$  of  $\text{BD}'_{r,s}$  generated by the elements  $\mathbf{c}_k^2 + 1, \mathbf{c}_l^2 - 1$ , and  $\mathbf{c}_i \mathbf{c}_j + \mathbf{c}_j \mathbf{c}_i$  in (2.2).

*Proof.* It suffices to show that  $\mathbf{e} \mathbf{c}_k^2 f + e f$ ,  $\mathbf{e} \mathbf{c}_l^2 f - e f$ , and  $\mathbf{e} \mathbf{c}_i \mathbf{c}_j f + \mathbf{e} \mathbf{c}_j \mathbf{c}_i f$  belong to  $\mathbf{M}$  for any two bead diagrams  $e, f \in \text{BD}'_{r,s}$ .

(i) First, we consider  $\mathbf{e} \mathbf{c}_k^2 f + e f$ . In constructing the diagram  $\tilde{d}$  from a diagram  $d$ , we delete an even number of beads along each strand. Therefore,  $\widetilde{\mathbf{e} \mathbf{c}_k^2 f} = \widetilde{e f}$ .

If the product  $\mathbf{c}_k^2$  in  $\mathbf{e} \mathbf{c}_k^2 f$  creates beads of type I, then  $\rho_1(\mathbf{e} \mathbf{c}_k^2 f) = \rho_1(e f) + 1$ . In this case,  $c(\mathbf{e} \mathbf{c}_k^2 f) \equiv c(e f)$ ,  $\ell_1(\mathbf{e} \mathbf{c}_k^2 f) \equiv \ell_1(e f)$ , and  $p_1(\mathbf{e} \mathbf{c}_k^2 f) \equiv p_1(e f) \pmod{2}$ . The other values  $\ell_2(\mathbf{e} \mathbf{c}_k^2 f)$ ,  $\rho_2(\mathbf{e} \mathbf{c}_k^2 f)$ ,  $p_2(\mathbf{e} \mathbf{c}_k^2 f)$  are the same as  $\ell_2(e f)$ ,  $\rho_2(e f)$ ,  $p_2(e f)$ , respectively. Thus,  $\beta(\mathbf{e} \mathbf{c}_k^2 f) \equiv \beta(e f) + 1 \pmod{2}$ .

If the product  $\mathbf{c}_k^2$  in  $\mathbf{e} \mathbf{c}_k^2 f$  creates beads along a vertical strand on the right-hand side of the wall, then  $\rho_2(\mathbf{e} \mathbf{c}_k^2 f) = \rho_2(e f)$ . In this case,  $p_2(\mathbf{e} \mathbf{c}_k^2 f) \equiv p_2(e f) + 1$ ,  $\ell_2(\mathbf{e} \mathbf{c}_k^2 f) \equiv \ell_2(e f)$ ,  $c(\mathbf{e} \mathbf{c}_k^2 f) \equiv c(e f) \pmod{2}$ . The other values  $\ell_1, \rho_1, p_1$  remain the same for  $\mathbf{e} \mathbf{c}_k^2 f$  as for  $e f$ . Hence,  $\beta(\mathbf{e} \mathbf{c}_k^2 f) \equiv \beta(e f) + 1 \pmod{2}$ . The other cases can be checked in a similar manner.

As a consequence,

$$\begin{aligned}
 (2.8) \quad e\mathbf{c}_k^2 f + ef &= e\mathbf{c}_k^2 f - (-1)^{\beta(e\mathbf{c}_k^2 f)} \widetilde{e\mathbf{c}_k^2 f} + (-1)^{\beta(e\mathbf{c}_k^2 f)} \widetilde{e\mathbf{c}_k^2 f} + ef \\
 &= e\mathbf{c}_k^2 f - (-1)^{\beta(e\mathbf{c}_k^2 f)} \widetilde{e\mathbf{c}_k^2 f} + ef - (-1)^{\beta(ef)} \widetilde{ef} \in \mathbf{M}.
 \end{aligned}$$

(ii) To verify  $e\mathbf{c}_l^2 f - ef \in \mathbf{M}$ , we can show that  $\widetilde{e\mathbf{c}_l^2 f} = \widetilde{ef}$  and  $\beta(e\mathbf{c}_l^2 f) \equiv \beta(ef) \pmod{2}$  as in (i) and then apply a calculation similar to that in (2.8).

(iii) To argue that  $e\mathbf{c}_i\mathbf{c}_j f + e\mathbf{c}_j\mathbf{c}_i f \in \mathbf{M}$ , assume  $\mathbf{c}_i$  creates a bead indexed by  $a$  and  $\mathbf{c}_j$  a bead indexed by  $a+1$  in  $e\mathbf{c}_i\mathbf{c}_j f$ . If we switch the beads containing  $a$  and  $a+1$ , we obtain the bead diagram  $e\mathbf{c}_j\mathbf{c}_i f$ . Since the number of beads along each strand does not change,  $\widetilde{e\mathbf{c}_i\mathbf{c}_j f} = \widetilde{e\mathbf{c}_j\mathbf{c}_i f}$ .

We will show that  $\beta(e\mathbf{c}_i\mathbf{c}_j f) \equiv \beta(e\mathbf{c}_j\mathbf{c}_i f) + 1 \pmod{2}$ . Suppose the beads created by  $\mathbf{c}_i$  and  $\mathbf{c}_j$  are of different types, say type I for  $\mathbf{c}_i$  and type II for  $\mathbf{c}_j$ . Then  $c(e\mathbf{c}_i\mathbf{c}_j f) = c(e\mathbf{c}_j\mathbf{c}_i f) - 1$ . The other values  $\ell_1, \ell_2, \rho_1, \rho_2, p_1, p_2$ , and  $\alpha$  are the same in  $e\mathbf{c}_i\mathbf{c}_j f$  and  $e\mathbf{c}_j\mathbf{c}_i f$ . Hence,  $\beta(e\mathbf{c}_i\mathbf{c}_j f) \equiv \beta(e\mathbf{c}_j\mathbf{c}_i f) + 1 \pmod{2}$ . The reverse possibility ( $\mathbf{c}_i$  type II and  $\mathbf{c}_j$  type I) can be treated similarly.

Now assume both  $\mathbf{c}_i$  and  $\mathbf{c}_j$  create beads of type I. If the beads are on the same strand, then

$$p_1(e\mathbf{c}_i\mathbf{c}_j f) \equiv p_1(e\mathbf{c}_j\mathbf{c}_i f) + 1 \pmod{2},$$

and the other values do not change. If they lie on different strands, then  $\ell_1(e\mathbf{c}_i\mathbf{c}_j f) \equiv \ell_1(e\mathbf{c}_j\mathbf{c}_i f) + 1 \pmod{2}$ , and the other values are unchanged. Therefore,  $\beta(e\mathbf{c}_i\mathbf{c}_j f) \equiv \beta(e\mathbf{c}_j\mathbf{c}_i f) + 1 \pmod{2}$ . The case that  $\mathbf{c}_i$  and  $\mathbf{c}_j$  create beads of type II can be handled similarly.

Applying a computation like the one in (2.8), we obtain  $e\mathbf{c}_i\mathbf{c}_j f + e\mathbf{c}_j\mathbf{c}_i f \in \mathbf{M}$ .  $\square$

We now prove the main theorem of this section.

**Theorem 2.9.** The walled Brauer-Clifford superalgebra  $\mathbf{BC}_{r,s}$  is isomorphic to the  $(r,s)$ -bead diagram algebra  $\mathbf{BD}_{r,s}$  as associative superalgebras.

*Proof.* Using the defining relations, we see that the linear map  $\phi_{r,s} : \mathbf{BC}_{r,s} \rightarrow \mathbf{BD}_{r,s}$  specified by

$$(2.10) \quad s_i \mapsto \mathbf{s}_i, \quad s_j \mapsto \mathbf{s}_j, \quad e_{r,r+1} \mapsto \mathbf{e}_{r,r+1}, \quad c_k \mapsto \mathbf{c}_k, \quad \text{and} \quad c_l \mapsto \mathbf{c}_l,$$

is a well-defined superalgebra epimorphism.

Recall that  $\mathbf{BD}_{r,s} := \mathbf{BD}'_{r,s}/\mathbf{L}$ , where  $\mathbf{L}$  is the two-sided ideal generated by the elements in (2.2). As  $\mathbf{L} \subseteq \mathbf{M}$ , by Lemma 2.7, there is well-defined linear map  $\pi_{r,s} : \mathbf{BD}_{r,s} \rightarrow \mathbf{BD}'_{r,s}/\mathbf{M}$  such that  $\pi_{r,s}(d + \mathbf{L}) = d + \mathbf{M}$  for  $d \in \mathbf{BD}'_{r,s}$ . Since  $\{c_P e_{p_1, q_1} \cdots e_{p_a, q_a} \sigma c_Q\}$  is a basis of  $\mathbf{BC}_{r,s}$  by Theorem 1.4 (iii), we can define a linear map  $\psi_{r,s} : \mathbf{BC}_{r,s} \rightarrow \mathbf{BD}'_{r,s}/\mathbf{M}$  such that

$$\psi_{r,s}(c_P e_{p_1, q_1} \cdots e_{p_a, q_a} \sigma c_Q) = c_P e_{p_1, q_1} \cdots e_{p_a, q_a} \sigma c_Q + \mathbf{M}.$$

Moreover,  $\pi_{r,s} \circ \phi_{r,s} = \psi_{r,s}$ . Therefore, if we can show that  $\psi_{r,s}$  is injective, it will follow that  $\phi_{r,s}$  is injective (hence, an isomorphism).

By Lemma 2.5 (ii), when  $\gamma(d) = 0$  for a bead diagram  $d$ , then  $d - (-1)^{\beta(d)} \widetilde{d} = 0$ . Thus,  $\mathbf{M}$  is spanned by the elements  $d - (-1)^{\beta(d)} \widetilde{d}$  with  $\gamma(d) \geq 1$ . Note that

$$\gamma(c_P e_{p_1, q_1} \cdots e_{p_a, q_a} \sigma c_Q) = 0.$$

Therefore, the set  $\{c_P e_{p_1, q_1} \cdots e_{p_a, q_a} \sigma c_Q + \mathbf{M}\}$  of these elements is linearly independent in  $\mathbf{BD}'_{r,s}/\mathbf{M}$ , so  $\psi_{r,s}$  is indeed injective.  $\square$

**Corollary 2.11.** The relation  $\mathbf{L} = \mathbf{M}$  holds. In particular,

$$(2.12) \quad \{d - (-1)^{\beta(d)} \widetilde{d} \mid d \text{ is a bead diagram in } \mathbf{BD}'_{r,s}, \gamma(d) \geq 1\}$$

is a basis of the two-sided ideal  $\mathbf{L}$ .

*Proof.* By Lemma 2.7 we have that  $\mathbf{L} \subseteq \mathbf{M}$ . Since the mapping  $\phi_{r,s}$  in (2.10) is an isomorphism and  $\pi_{r,s} = \psi_{r,s} \circ \phi_{r,s}^{-1}$ , we know that  $\pi_{r,s}$  is injective. Thus, for  $m \in \mathbf{M}$ ,  $\pi_{r,s}(m + \mathbf{L}) = 0 + \mathbf{M}$  implies that  $m \in \mathbf{L}$ .

From the proof of Theorem 2.9, we have that the set in (2.12) spans  $\mathbf{M} (= \mathbf{L})$ . Since  $\gamma(d) \geq 1$  and  $\gamma(\widetilde{d}) = 0$ , it follows that (2.12) is a linearly independent set.  $\square$

### 3. THE QUANTUM WALLED BRAUER-CLIFFORD SUPERALGEBRA $\mathbf{BC}_{r,s}(q)$

Let  $q$  be an indeterminate and  $\mathbb{C}(q)$  be the field of rational functions in  $q$ . Set  $\mathbf{V}_q = \mathbb{C}(q) \otimes_{\mathbb{C}} \mathbf{V} = \mathbb{C}(q) \otimes_{\mathbb{C}} \mathbb{C}(n|n)$ . Corresponding to any  $X = \sum_k Y_k \otimes Z_k \in (\text{End}_{\mathbb{C}(q)}(\mathbf{V}_q))^{\otimes 2}$ , let  $X^{12} = \sum_k Y_k \otimes Z_k \otimes \text{id}$ ,  $X^{13} = \sum_k Y_k \otimes \text{id} \otimes Z_k$ , and  $X^{23} = \sum_k \text{id} \otimes Y_k \otimes Z_k$  in  $(\text{End}_{\mathbb{C}(q)}(\mathbf{V}_q))^{\otimes 3}$ , where  $\text{id} = \text{id}_{\mathbf{V}_q}$ .

Let  $\xi = q - q^{-1}$  and define  $S = \sum_{i,j \in \mathbf{I}} S_{ij} \otimes E_{ij} \in (\text{End}_{\mathbb{C}(q)}(\mathbf{V}_q))^{\otimes 2}$  by

$$(3.1) \quad S = \sum_{i,j \in \mathbf{I}} q^{(\delta_{ij} + \delta_{i,-j})(1-2|j|)} E_{ii} \otimes E_{jj} + \xi \sum_{i,j \in \mathbf{I}, i < j} (-1)^{|i|} (E_{ji} + E_{-j,-i}) \otimes E_{ij}.$$

$S$  is known to satisfy the quantum Yang-Baxter equation  $S^{12}S^{13}S^{23} = S^{23}S^{13}S^{12}$ . In [18], Olshanski constructed a quantization of  $\mathfrak{U}(\mathfrak{q}(n))$  of  $\mathfrak{q}(n)$  in terms of  $S$ .

**Definition 3.2.** [18] The *quantum queer superalgebra*  $\mathfrak{U}_q(\mathfrak{q}(n))$  is the unital associative superalgebra over  $\mathbb{C}(q)$  generated by elements  $u_{ij}$  with  $i \leq j$  and  $i, j \in \mathbf{I} = \{\pm i \mid i = 1, \dots, n\}$ , which satisfy the following relations:

$$(3.3) \quad u_{ii}u_{-i,-i} = 1 = u_{-i,-i}u_{ii}, \quad U^{12}U^{13}S^{23} = S^{23}U^{13}U^{12},$$

where  $U = \sum_{i,j \in \mathbf{I}, i \leq j} u_{ij} \otimes E_{ij}$ , and the last equality holds in  $\mathfrak{U}_q(\mathfrak{q}(n)) \otimes_{\mathbb{C}(q)} (\text{End}_{\mathbb{C}(q)}(\mathbf{V}_q))^{\otimes 2}$ . The  $\mathbb{Z}_2$ -degree of  $u_{ij}$  is  $|i| + |j|$ .

By the construction, the assignment  $u_{ij} \mapsto S_{ij}$  is a representation of  $\mathfrak{U}_q(\mathfrak{q}(n))$  on  $\mathbf{V}_q$  (see [18, Sec. 4]). The superalgebra  $\mathfrak{U}_q(\mathfrak{q}(n))$  is a Hopf superalgebra with coproduct  $\Delta(U) = U^{13}U^{23} \in (\mathfrak{U}_q(\mathfrak{q}(n)))^{\otimes 2} \otimes_{\mathbb{C}(q)} \text{End}_{\mathbb{C}(q)}(\mathbf{V}_q)$ , or more explicitly,  $\Delta(u_{ij}) = \sum_{k \in \mathbf{I}} (-1)^{(|i|+|k|)(|k|+|j|)} u_{ik} \otimes u_{kj}$ . The counit is given by  $\varepsilon(U) = 1$  and the antipode by  $U \mapsto U^{-1}$ .

Let  $\mathbf{V}_q^{r,s} := \mathbf{V}_q^{\otimes r} \otimes_{\mathbb{C}(q)} (\mathbf{V}_q^*)^{\otimes s}$  be the mixed tensor space of  $\mathbf{V}_q$  and  $\mathbf{V}_q^*$ . Then  $\mathbf{V}_q^{r,s}$  is a representation of  $\mathfrak{U}_q(\mathfrak{q}(n))$  via the coproduct and antipode mappings. To describe the structure of the centralizer superalgebra  $\text{End}_{\mathfrak{U}_q(\mathfrak{q}(n))}(\mathbf{V}_q^{r,s})$ , we introduce the *quantum walled Brauer-Clifford superalgebra*  $\mathbf{BC}_{r,s}(q)$ .

**Definition 3.4.** The *quantum walled Brauer-Clifford superalgebra*  $\mathbf{BC}_{r,s}(q)$  is the associative superalgebra over  $\mathbb{C}(q)$  generated by even elements  $t_1, t_2, \dots, t_{r-1}, t_1^*, t_2^*, \dots, t_{s-1}^*$ ,  $e$  and odd elements

$c_1, c_2, \dots, c_r, c_1^*, c_2^*, \dots, c_s^*$  satisfying the following defining relations (for  $i, j$  in the allowable range):

$$\begin{aligned}
 (3.5) \quad & t_i^2 - (q - q^{-1})t_i - 1 = 0, \quad t_i t_{i+1} t_i = t_{i+1} t_i t_{i+1}, \quad (t_i^*)^2 - (q - q^{-1})t_i^* - 1 = 0, \quad t_i^* t_{i+1}^* t_i^* = t_{i+1}^* t_i^* t_{i+1}^*, \\
 & t_i t_j = t_j t_i \quad (|i - j| > 1), \quad t_i t_j^* = t_j^* t_i, \quad t_i^* t_j^* = t_j^* t_i^* \quad (|i - j| > 1), \\
 & e^2 = 0, \quad et_{r-1}e = e, \quad et_j = t_j e \quad (j \neq r-1), \quad et_1^* e = e, \quad et_j^* = t_j^* e \quad (j \neq 1), \\
 & et_{r-1}^{-1} t_1^* et_1^* t_{r-1}^{-1} = t_{r-1}^{-1} t_1^* et_1^* t_{r-1}^{-1} e, \\
 & c_i^2 = -1, \quad c_i c_j = -c_j c_i \quad (i \neq j), \quad c_i c_j^* = -c_j^* c_i, \quad (c_i^*)^2 = 1, \quad c_i^* c_j^* = -c_j^* c_i^* \quad (i \neq j), \\
 & t_i c_i = c_{i+1} t_i, \quad t_i c_j = c_j t_i \quad (j \neq i, i+1), \quad t_i^* c_i^* = c_{i+1}^* t_i^*, \quad t_i^* c_j^* = c_j^* t_i^* \quad (j \neq i, i+1), \\
 & t_i c_j^* = c_j^* t_i, \quad c_r e = c_1^* e, \quad c_j e = ec_j \quad (j \neq r), \quad t_i^* c_j = c_j t_i^*, \quad ec_r = ec_1^*, \quad c_j^* e = ec_j^* \quad (j \neq 1), \\
 & ec_r e = 0.
 \end{aligned}$$

**Definition 3.6.** (i) The (finite) Hecke-Clifford superalgebra  $HC_r(q)$  in [18] is the associative superalgebra over  $\mathbb{C}(q)$  generated by the even elements  $t_1, t_2, \dots, t_{r-1}$  and the odd elements  $c_1, c_2, \dots, c_r$  with the following defining relations (for allowable  $i, j$ ):

$$\begin{aligned}
 (3.7) \quad & t_i^2 - (q - q^{-1})t_i - 1 = 0, \quad t_i t_{i+1} t_i = t_{i+1} t_i t_{i+1}, \\
 & t_i t_j = t_j t_i \quad (|i - j| > 1), \\
 & c_i^2 = -1, \quad c_i c_j = -c_j c_i \quad (i \neq j), \\
 & t_i c_i = c_{i+1} t_i, \quad t_i c_j = c_j t_i \quad (j \neq i, i+1).
 \end{aligned}$$

(ii) The quantum walled Brauer algebra  $H_{r,s}^0(q)$  in [11] is the associative algebra over  $\mathbb{C}(q)$  generated by the elements  $t_1, t_2, \dots, t_{r-1}, t_1^*, \dots, t_{s-1}^*$  and  $e$  which satisfy the first four lines in (3.5).

**Remark 3.8.** The relations in the first three lines in (3.5) appear in Definition 2.1 of [11]. In line 4 of (3.5), we have the one relation

$$(3.9) \quad et_{r-1}^{-1} t_1^* et_1^* t_{r-1}^{-1} = t_{r-1}^{-1} t_1^* et_1^* t_{r-1}^{-1} e.$$

instead of the following two relations of [11]:

$$(3.10) \quad et_{r-1}^{-1} t_1^* et_{r-1} = et_{r-1}^{-1} t_1^* et_1^*, \quad t_{r-1} et_{r-1}^{-1} t_1^* e = t_1^* et_{r-1}^{-1} t_1^* e.$$

The relations in (3.10) can be derived using (3.9) and various identities from (3.5) (especially the fact that  $t_{r-1}$  and  $t_1^*$  commute) in the following way:

$$\begin{aligned}
 et_{r-1}^{-1} t_1^* e &= (et_{r-1}^{-1} e) t_{r-1}^{-1} t_1^* e = e((t_1^*)^{-1} t_1^*) t_{r-1}^{-1} et_{r-1}^{-1} t_1^* e \\
 &= e(t_1^*)^{-1} (t_1^* t_{r-1}^{-1} et_{r-1}^{-1} t_1^* e) = e(t_1^*)^{-1} (et_1^* t_{r-1}^{-1} et_{r-1}^{-1} t_1^*) \\
 &= (e(t_1^*)^{-1} e) t_1^* t_{r-1}^{-1} et_{r-1}^{-1} t_1^* = et_1^* t_{r-1}^{-1} et_{r-1}^{-1} t_1^* = et_{r-1}^{-1} t_1^* et_1^* t_{r-1}^{-1} = t_{r-1}^{-1} t_1^* et_{r-1}^{-1} t_1^* e,
 \end{aligned}$$

implying both relations in (3.10).

The following simple expression will be useful in several calculations henceforth.

**Lemma 3.11.** With the conventions  $(-1)^{|0|} = 0$ ,  $|j| = 1$  for any integer  $j < 0$ , and  $|j| = 0$  for  $j > 0$ , we have

$$(3.12) \quad \xi \sum_{i < j < k} (-1)^{|j|} q^{2j(1-2|j|)} = q^{(2k-1)(1-2|k|)} - q^{(2i+1)(1-2|i|)}$$

for any nonzero integers  $i < k$ , where  $\xi = q - q^{-1}$  as above.

*Proof.* This can be checked by considering the three cases  $0 < i < k$ ,  $i < k < 0$  and  $i < 0 < k$ .  $\square$

In order to construct an action of  $\text{BC}_{r,s}(q)$  on the mixed tensor space  $\mathbf{V}_q^{r,s}$ , we will need a number of  $\mathfrak{U}_q(\mathfrak{q}(n))$ -module homomorphisms. Note that  $\mathbb{C}(q)$  becomes a  $\mathfrak{U}_q(\mathfrak{q}(n))$ -module by sending  $U$  to the identity map in  $\text{End}_{\mathbb{C}(q)}(\mathbb{C}(q) \otimes_{\mathbb{C}(q)} \mathbf{V}_q)$ .

**Lemma 3.13.** There are  $\mathfrak{U}_q(\mathfrak{q}(n))$ -module homomorphisms  $\cap : \mathbb{C}(q) \rightarrow \mathbf{V}_q \otimes \mathbf{V}_q^*$  and  $\cup : \mathbf{V}_q \otimes \mathbf{V}_q^* \rightarrow \mathbb{C}(q)$  given by

$$\cap(1) = \sum_{i \in \mathbf{I}} v_i \otimes \omega_i, \quad \cup(v_i \otimes \omega_j) = (-1)^{|i|} q^{2i(1-2|i|)-(2n+1)} \delta_{ij}.$$

*Proof.* Since  $\cap$  is the canonical map  $\mathbb{C}(q) \rightarrow \mathbf{V}_q \otimes \mathbf{V}_q^*$ , we have

$$(3.14) \quad (X \otimes \text{id})\cap = (\text{id} \otimes X^T)\cap$$

for any  $X \in \text{End}_{\mathbb{C}(q)}(\mathbf{V}_q)$ , where  $^T$  denotes the supertranspose. Since  $\cap$  is even, it follows that

$$((S^{23})^{-1})^{T_2} (\cap \otimes \text{id}) = (S^{13})^{-1} (\cap \otimes \text{id})$$

in  $\text{Hom}_{\mathbb{C}(q)}(\mathbb{C}(q) \otimes \mathbf{V}_q, \mathbf{V}_q \otimes \mathbf{V}_q^* \otimes \mathbf{V}_q)$ , where  $^{T_2}$  indicates taking the supertranspose on the second factor. Thus

$$S^{13} ((S^{23})^{-1})^{T_2} (\cap \otimes \text{id}) = (\cap \otimes \text{id}).$$

The action of  $\mathfrak{U}_q(\mathfrak{q}(n))$  on  $\mathbf{V}_q \otimes \mathbf{V}_q^*$  (resp. on  $\mathbb{C}(q)$ ) is defined by sending  $U$  to  $S^{13} ((S^{23})^{-1})^{T_2}$  (resp. to  $\text{id}$ ), so this shows that  $\cap$  is a  $\mathfrak{U}_q(\mathfrak{q}(n))$ -module homomorphism.

To check that  $\cup$  is a homomorphism, we require an explicit expression for  $S^{13} ((S^{23})^{-1})^{T_2}$ . We have

$$S^{-1} = \sum_{i,j \in \mathbf{I}} q^{-(\delta_{ij} + \delta_{i,-j})(1-2|j|)} E_{ii} \otimes E_{jj} - \xi \sum_{i,j \in \mathbf{I}, i < j} (-1)^{|i|} (E_{ji} + E_{-j,-i}) \otimes E_{ij}.$$

If we identify  $\mathbf{V}_q$  with  $\mathbf{V}_q^*$  via  $v_i \mapsto \omega_i$ , then  $(S^{-1})^{T_1}$  becomes identified with an endomorphism  $S^*$  of  $\mathbf{V}_q \otimes \mathbf{V}_q$  given by

$$(3.15) \quad S^* = \sum_{i,j \in \mathbf{I}} q^{-(\delta_{ij} + \delta_{i,-j})(1-2|j|)} E_{ii} \otimes E_{jj} - \xi \sum_{i,j \in \mathbf{I}, i < j} (-1)^{|i||j|} ((-1)^{|i|+|j|} E_{ij} + E_{-i,-j}) \otimes E_{ij}.$$

Therefore, identifying  $\mathbf{V}_q \otimes \mathbf{V}_q^*$  with  $\mathbf{V}_q \otimes \mathbf{V}_q$ , we have that the action on  $\mathbf{V}_q \otimes \mathbf{V}_q$  is defined by sending  $U$  to

$$\begin{aligned} S^{13}(S^*)^{23} &= \sum_{i,j,k \in \mathbf{I}} q^{(\delta_{ij} + \delta_{i,-j} - \delta_{jk} - \delta_{j,-k})(1-2|j|)} E_{ii} \otimes E_{kk} \otimes E_{jj} \\ &\quad - \xi \sum_{i \in \mathbf{I}} \sum_{j,k \in \mathbf{I}, j < k} (-1)^{|k||j|} q^{(\delta_{ij} + \delta_{i,-j})(1-2|j|)} E_{ii} \otimes ((-1)^{|j|+|k|} E_{jk} + E_{-j,-k}) \otimes E_{jk} \\ &\quad + \xi \sum_{j,k \in \mathbf{I}, j < k} \sum_{i \in \mathbf{I}} (-1)^{|j|} q^{(\delta_{ik} + \delta_{i,-k})(1-2|k|)} (E_{kj} + E_{-k,-j}) \otimes E_{ii} \otimes E_{jk} \\ &\quad - \xi^2 \sum_{i,j,k \in \mathbf{I}, j < i < k} (-1)^{|i||j|+|j||k|+|j|} (E_{ij} + E_{-i,-j}) \otimes ((-1)^{|k|} E_{ik} + (-1)^{|i|} E_{-i,-k}) \otimes E_{jk}. \end{aligned}$$

The map  $\cup$  can be identified with the map  $q^{-(2n+1)} \sum_{i \in \mathbf{I}} q^{2i(1-2|i|)} \omega_i \otimes \omega_i$ , and  $\omega_k E_{ij} = \delta_{k,i} \omega_j$ . Moreover, direct calculations show

$$\begin{aligned}
& q^{2n+1}(\cup \otimes \text{id}) S^{13}(S^*)^{23} - q^{2n+1}(\cup \otimes \text{id}) \\
&= -\xi \sum_{j < k} (-1)^{|k||j|} q^{(1-2|j|)} q^{2j(1-2|j|)} ((-1)^{|j|+|k|} \omega_j \otimes \omega_k + \omega_{-j} \otimes \omega_{-k}) \otimes E_{jk} \\
&+ \xi \sum_{j < k} (-1)^{|j|} q^{-(1-2|k|)} q^{2k(1-2|k|)} ((-1)^{|k|(|k|+|j|)} \omega_j \otimes \omega_k + (-1)^{(|k|+1)(|k|+|j|)} \omega_{-j} \otimes \omega_{-k}) \otimes E_{jk} \\
&- \xi^2 \sum_{j < i < k} (-1)^{|i||j|+|j||k|+|j|} q^{2i(1-2|i|)} ((-1)^{|k|+|i|(|i|+|j|)} \omega_j \otimes \omega_k \\
&\quad + (-1)^{|i|+(|i|+1)(|i|+|j|)} \omega_{-j} \otimes \omega_{-k}) \otimes E_{jk} \\
&= \xi \sum_{j < k} \left( q^{(2k-1)(1-2|k|)} - q^{(2j+1)(1-2|j|)} - \xi \sum_{k > i > j} (-1)^{|i|} q^{2i(1-2|i|)} \right) \\
&\quad \left( (-1)^{|k||j|+|k|+|j|} \omega_j \otimes \omega_k + (-1)^{|k||j|} \omega_{-j} \otimes \omega_{-k} \right) \otimes E_{jk} \\
&= 0 \text{ by (3.12).}
\end{aligned}$$

Therefore  $(\cup \otimes \text{id}) S^{13}(S^*)^{23} = (\cup \otimes \text{id})$ , so  $\cup$  defines a  $\mathfrak{U}_q(\mathfrak{q}(n))$ -module homomorphism.  $\square$

**Theorem 3.16.** There is an action of  $\text{BC}_{r,s}(q)$  on  $\mathbf{V}_q^{r,s}$  which supercommutes with the action of  $\mathfrak{U}_q(\mathfrak{q}(n))$ , such that the action of each generator is given by

$$\begin{aligned}
\mathbf{t}_i &\mapsto \text{id}^{\otimes(i-1)} \otimes PS \otimes \text{id}^{\otimes(r+s-1-i)}, & \mathbf{c}_i &\mapsto \text{id}^{\otimes(i-1)} \otimes \Phi \otimes \text{id}^{\otimes(r+s-i)}, \\
\mathbf{t}_i^* &\mapsto \text{id}^{\otimes(r+i-1)} \otimes P^T S^T \otimes \text{id}^{\otimes(s-1-i)}, & \mathbf{c}_i^* &\mapsto \text{id}^{\otimes(r+i-1)} \otimes \Phi^T \otimes \text{id}^{\otimes(s-i)}, \\
\mathbf{e} &\mapsto \text{id}^{\otimes(r-1)} \otimes \cap \cup \otimes \text{id}^{\otimes(s-1)},
\end{aligned}$$

where  $\text{id} = \text{id}_{\mathbf{V}_q}$ ,

$$(3.17) \quad P = \sum_{i,j \in \mathbf{I}} (-1)^{|j|} E_{ij} \otimes E_{ji} \in \text{End}(\mathbf{V}_q \otimes \mathbf{V}_q), \quad \Phi = \sum_{i \in \mathbf{I}} (-1)^{|i|} E_{i,-i} \in \text{End}(\mathbf{V}_q),$$

and  $^T$  is the supertranspose. Explicitly, identifying  $\mathbf{V}_q$  with  $\mathbf{V}_q^*$  as above, we have

$$\begin{aligned}
(3.18) \quad PS &= \sum_{i,j \in \mathbf{I}} (-1)^{|i|} q^{(\delta_{ij} + \delta_{-i,j})(1-2|j|)} E_{ji} \otimes E_{ij} \\
&+ \xi \sum_{i,j \in \mathbf{I}, i < j} (E_{ii} \otimes E_{jj} - (-1)^{|i|+|j|} E_{i,-i} \otimes E_{-j,j}), \\
P^T S^T &= \sum_{i,j \in \mathbf{I}} (-1)^{|i|} q^{(\delta_{ij} + \delta_{-i,j})(1-2|j|)} E_{ji} \otimes E_{ij} + \xi \sum_{i,j \in \mathbf{I}, j < i} (E_{ii} \otimes E_{jj} - E_{i,-i} \otimes E_{-j,j}), \\
\Phi^T &= \sum_{i \in \mathbf{I}} E_{i,-i}, \quad \cap \cup = q^{-(2n+1)} \sum_{i,j \in \mathbf{I}} (-1)^{|i||j|} q^{2j(1-2|j|)} E_{ij} \otimes E_{ij}.
\end{aligned}$$

*Proof.* The fact that the actions of  $\mathbf{t}_i$  and  $\mathbf{c}_i$  are  $\mathfrak{U}_q(\mathfrak{q}(n))$ -module endomorphisms satisfying the relations of  $\text{HC}_r(q)$  is shown in [18]. Consider the linear map given by the cyclic permutation  $\sigma : \mathbf{V}_q^{\otimes s} \rightarrow \mathbf{V}_q^{\otimes s}$ ,  $v_1 \otimes \cdots \otimes v_s \mapsto (-1)^{\sum_{i < j} |v_i||v_j|} v_s \otimes \cdots \otimes v_1$ . Conjugating by  $\sigma$ , we obtain another action of  $\text{HC}_s(q)$  on  $\mathbf{V}_q^{\otimes s}$  specified by

$$\mathbf{t}_i \mapsto \text{id}^{\otimes(s-1-i)} \otimes SP \otimes \text{id}^{\otimes(i-1)}, \quad \mathbf{c}_i \mapsto \text{id}^{\otimes(s-i)} \otimes \Phi \otimes \text{id}^{\otimes(i-1)}.$$

These maps are also  $\mathfrak{U}_q(\mathfrak{q}(n))$ -module endomorphisms (even though  $\sigma$  is not). Applying the anti-automorphism of  $\mathbf{HC}_s(q)$  that sends  $\mathbf{t}_i$  to  $\mathbf{t}_{s-i}$  and  $\mathbf{c}_i$  to  $\mathbf{c}_{s+1-i}$ , we see that

$$\mathbf{t}_i^* \mapsto \text{id}^{\otimes(i-1)} \otimes P^T S^T \otimes \text{id}^{\otimes(s-1-i)}, \quad \mathbf{c}_i^* \mapsto \text{id}^{\otimes(i-1)} \otimes \Phi^T \otimes \text{id}^{\otimes(s-i)}$$

satisfy the required relations.

Since  $\cap$  and  $\cup$  are  $\mathfrak{U}_q(\mathfrak{q}(n))$ -module homomorphisms, the same is true of  $\mathbf{e}$ . We have

$$\cup(\text{id} \otimes \Phi^T) = q^{-(2n+1)} \sum_{i \in \mathbf{I}} q^{2i(1-2|i|)} \omega_{-i} \otimes \omega_i = \cup(\Phi \otimes \text{id})$$

Thus,  $\cup(\text{id} \otimes \Phi^T) = \cup(\Phi \otimes \text{id})$ , so  $\mathbf{e} \mathbf{c}_r = \mathbf{e} \mathbf{c}_1^*$ . Also  $(\text{id} \otimes \Phi^T) \cap = (\Phi \otimes \text{id}) \cap$  by (3.14), so  $\mathbf{c}_r \mathbf{e} = \mathbf{c}_1^* \mathbf{e}$ . We have

$$\begin{aligned} \cup \cap (1) &= q^{-(2n+1)} \left( \sum_{j \in \mathbf{I}} q^{2j(1-2|j|)} \omega_j \otimes \omega_j \right) \left( \sum_{i \in \mathbf{I}} v_i \otimes v_i \right) \\ &= q^{-(2n+1)} \sum_{i \in \mathbf{I}} (-1)^{|i|} q^{2i(1-2|i|)} = 0 \\ \cup(\Phi \otimes \text{id}) \cap (1) &= q^{-(2n+1)} \left( \sum_{i \in \mathbf{I}} q^{2i(1-2|i|)} \omega_{-i} \otimes \omega_i \right) \left( \sum_{j \in \mathbf{I}} v_j \otimes v_j \right) = 0, \end{aligned}$$

so  $\mathbf{e}^2 = \mathbf{e} \mathbf{c}_r \mathbf{e} = 0$ . Now using  $q^{2n+1} \cup (E_{ij} \otimes \text{id}) \cap (1) = (-1)^{|i|} q^{2i(1-2|i|)} \delta_{ij}$  and identifying  $\mathbf{V}_q$  with  $\mathbf{V}_q \otimes \mathbb{C}(q)$ , we have

$$\begin{aligned} q^{2n+1} (\text{id} \otimes \cup) (PS \otimes \text{id}) (\text{id} \otimes \cap) &= \sum_{j \in \mathbf{I}} q^{(2j+1)(1-2|j|)} E_{jj} + \xi \sum_{i, j \in \mathbf{I}, j < i} (-1)^{|i|} q^{2i(1-2|i|)} E_{jj} \\ &= \sum_{j \in \mathbf{I}} \left( q^{(2j+1)(1-2|j|)} + \xi \sum_{i, j \in \mathbf{I}, j < i} (-1)^{|i|} q^{2i(1-2|i|)} \right) E_{jj} \\ &= q^{2n+1} \text{id} \quad \text{by (3.12).} \end{aligned}$$

Thus  $(\text{id} \otimes \cup)(PS \otimes \text{id})(\text{id} \otimes \cap) = (\text{id} \otimes \cap)$ , so  $\mathbf{e} \mathbf{t}_{r-1} \mathbf{e} = \mathbf{e}$ . Similarly,

$$q^{2n+1} (\cup \otimes \text{id}) (\text{id} \otimes P^T S^T) (\cap \otimes \text{id}) = \sum_j q^{(2j+1)(1-2|j|)} E_{jj} + \xi \sum_{i > j} (-1)^{|i|} q^{2i(1-2|i|)} E_{jj} = q^{2n+1} \text{id}.$$

Thus  $(\cap \cup \otimes \text{id})(\text{id} \otimes P^T S^T)(\cap \cup \otimes \text{id}) = (\cap \cup \otimes \text{id})$ , so  $\mathbf{e} \mathbf{t}_1^* \mathbf{e} = \mathbf{e}$ .

Identifying  $\mathbf{V}_q \otimes \mathbf{V}_q^*$  with  $\mathbf{V}_q \otimes \mathbb{C}(q) \otimes \mathbf{V}_q^*$ , we have

$$\begin{aligned} (P \otimes \text{id}_{\mathbf{V}_q^* \otimes \mathbf{V}_q^*}) (\text{id}_{\mathbf{V}_q} \otimes \cap \otimes \text{id}_{\mathbf{V}_q^*}) \cap (1) &= (P \otimes \text{id}) \left( \sum_{i, j} v_j \otimes v_i \otimes \omega_i \otimes \omega_j \right) \\ &= \sum_{i, j} (-1)^{|i||j|} v_i \otimes v_j \otimes \omega_i \otimes \omega_j. \end{aligned}$$

The above is the canonical map  $\mathbb{C}(q) \rightarrow \mathbf{V}_q \otimes \mathbf{V}_q \otimes \mathbf{V}_q^* \otimes \mathbf{V}_q^*$ , so by the same reasoning that led to (3.14), we know

$$(SP \otimes \text{id})(P \otimes \text{id})(\text{id} \otimes \cap \otimes \text{id}) \cap = (\text{id} \otimes P^T S^T)(P \otimes \text{id})(\text{id} \otimes \cap \otimes \text{id}) \cap.$$

Thus

$$(3.19) \quad (PS \otimes \text{id})(\text{id} \otimes \cap \otimes \text{id}) \cap = (\text{id} \otimes P^T S^T)(\text{id} \otimes \cap \otimes \text{id}) \cap.$$



To prove the corresponding expression for  $\cup$ , we must explicitly compute the following:

$$\begin{aligned}
& q^{4n+2} \cup (\text{id} \otimes \cup \otimes \text{id})(PS \otimes \text{id}) \\
&= q^{4n+2} \cup (\text{id} \otimes \cup \otimes \text{id}) \left( \sum_{i,j} (-1)^{|i|} q^{(\delta_{ij} + \delta_{-i,j})(1-2|j|)} E_{ji} \otimes E_{ij} \otimes \text{id} \otimes \text{id} \right. \\
&\quad \left. + \xi \sum_{i>j} (E_{jj} \otimes E_{ii} - (-1)^{|i|+|j|} E_{j,-j} \otimes E_{-i,i}) \otimes \text{id} \otimes \text{id} \right) \\
&= \sum_{i,j} (-1)^{|i||j|} q^{(\delta_{ij} + \delta_{-i,j} + 2j)(1-2|j|) + 2i(1-2|i|)} \omega_i \otimes \omega_j \otimes \omega_i \otimes \omega_j \\
&\quad + \xi \sum_{i>j} q^{2i(1-2|i|) + 2j(1-2|j|)} \left( \omega_j \otimes \omega_i \otimes \omega_i \otimes \omega_j + (-1)^{|j|} \omega_{-j} \otimes \omega_i \otimes \omega_{-i} \otimes \omega_j \right),
\end{aligned}$$

$$\begin{aligned}
& q^{4n+2} \cup (\text{id} \otimes \cup \otimes \text{id})(\text{id} \otimes P^T S^T) \\
&= q^{4n+2} \cup (\text{id} \otimes \cup \otimes \text{id}) \left( \sum_{i,j} (-1)^{|i|} q^{(\delta_{ij} + \delta_{-i,j})(1-2|j|)} \text{id} \otimes \text{id} \otimes E_{ji} \otimes E_{ij} \right. \\
&\quad \left. + \xi \sum_{i>j} \text{id} \otimes \text{id} \otimes (E_{ii} \otimes E_{jj} - E_{i,-i} \otimes E_{-j,j}) \right) \\
&= q^{2n+1} \cup \left( \sum_{i,j} (-1)^{|i|} q^{(\delta_{ij} + \delta_{-i,j} + 2j)(1-2|j|)} \text{id} \otimes \omega_j \otimes \omega_i \otimes E_{ij} \right. \\
&\quad \left. + \xi \sum_{i>j} q^{2i(1-2|i|)} (\text{id} \otimes \omega_i \otimes \omega_i \otimes E_{jj} - \text{id} \otimes \omega_i \otimes \omega_{-i} \otimes E_{-j,j}) \right) \\
&= q^{4n+2} \cup (\text{id} \otimes \cup \otimes \text{id})(PS \otimes \text{id}).
\end{aligned}$$

Thus,

$$(3.20) \quad \cup (\text{id} \otimes \cup \otimes \text{id})(PS \otimes \text{id}) = \cup (\text{id} \otimes \cup \otimes \text{id})(\text{id} \otimes P^T S^T).$$

Finally, using

$$S^{-1}P = \sum_{i,j} (-1)^{|j|} q^{-(\delta_{ij} + \delta_{-i,j})(1-2|j|)} E_{ij} \otimes E_{ji} - \xi \sum_{i>j} \left( E_{ii} \otimes E_{jj} + (-1)^{|i|+|j|} E_{-i,i} \otimes E_{j,-j} \right)$$

and

$$q^{2n+1} \cup (E_{ij} \otimes E_{kl}) \cap = \delta_{ik} \delta_{jl} (-1)^{|i||j|+|i|+|j|} q^{2i(1-2|i|)},$$

we have (with the help of (3.12))

$$\begin{aligned}
& q^{2n+1} (\text{id} \otimes \cup \otimes \text{id})(S^{-1}P \otimes P^T S^T)(\text{id} \otimes \cap \otimes \text{id}) \\
&= \sum_{i,j} q^{2j(1-2|j|)} (-1)^{|i||j|} E_{ij} \otimes E_{ij} \\
&\quad + \xi \sum_{i>j} (q^{(2i-1)(1-2|i|)} - q^{(2j+1)(1-2|j|)}) \left( E_{ii} \otimes E_{jj} + (-1)^{|i|} E_{-i,i} \otimes E_{-j,j} \right) \\
&\quad - \xi^2 \sum_{i>k>j} (-1)^{|k|} q^{2k(1-2|k|)} \left( E_{ii} \otimes E_{jj} + (-1)^{|i|} E_{-i,i} \otimes E_{-j,j} \right) \\
&= q^{2n+1} \cap \cup.
\end{aligned}$$

Combining this with (3.20) gives

$$\begin{aligned} & (\text{id} \otimes \cap \cup \otimes \text{id})(S^{-1}P \otimes P^T S^T)(\text{id} \otimes \cap \cup \otimes \text{id})(PS \otimes \text{id}) \\ &= (\text{id} \otimes \cap \otimes \text{id}) \cap \cup (\text{id} \otimes \cup \otimes \text{id})(PS \otimes \text{id}) \\ &= (\text{id} \otimes \cap \cup \otimes \text{id})(S^{-1}P \otimes P^T S^T)(\text{id} \otimes \cap \cup \otimes \text{id})(\text{id} \otimes P^T S^T). \end{aligned}$$

Thus,  $\text{et}_{r-1}^{-1} \mathbf{t}_1^* \text{et}_{r-1} = \text{et}_{r-1}^{-1} \mathbf{t}_1^* \text{et}_1^*$ . Similarly combining with (3.19) shows that

$$\begin{aligned} & (PS \otimes \text{id})(\text{id} \otimes \cap \cup \otimes \text{id})(S^{-1}P \otimes P^T S^T)(\text{id} \otimes \cap \cup \otimes \text{id}) \\ &= (PS \otimes \text{id})(\text{id} \otimes \cap \otimes \text{id}) \cap \cup (\text{id} \otimes \cup \otimes \text{id}) \\ &= (\text{id} \otimes P^T S^T)(\text{id} \otimes \cap \cup \otimes \text{id})(S^{-1}P \otimes P^T S^T)(\text{id} \otimes \cap \cup \otimes \text{id}). \end{aligned}$$

Hence,  $\mathbf{t}_{r-1} \text{et}_{r-1}^{-1} \mathbf{t}_1^* \mathbf{e} = \mathbf{t}_1^* \text{et}_{r-1}^{-1} \mathbf{t}_1^* \mathbf{e}$ , and it follows that  $\text{et}_{r-1}^{-1} \mathbf{t}_1^* \text{et}_1^* \text{et}_{r-1}^{-1} = \mathbf{t}_{r-1}^{-1} \mathbf{t}_1^* \text{et}_{r-1}^{-1} \mathbf{t}_1^* \mathbf{e}$ .  $\square$

**Proposition 3.21.** The walled Brauer-Clifford superalgebra  $\text{BC}_{r,s}$  is the *classical limit* of the quantum walled Brauer-Clifford superalgebra  $\text{BC}_{r,s}(q)$ .

*Proof.* To see this, let  $\mathcal{R} = \mathbb{C}[q, q^{-1}]_{(q-1)}$  be the localization of  $\mathbb{C}[q, q^{-1}]$  at the ideal generated by  $q-1$ . Let  $\text{BC}_{r,s}(\mathcal{R})$  be the  $\mathcal{R}$ -subalgebra of  $\text{BC}_{r,s}(q)$  generated by  $\mathbf{t}_1, \dots, \mathbf{t}_{r-1}, \mathbf{c}_1, \dots, \mathbf{c}_r, \mathbf{t}_1^*, \dots, \mathbf{t}_{s-1}^*, \mathbf{c}_1^*, \dots, \mathbf{c}_s^*, \mathbf{e}$ . Let  $\mathbf{V}_{\mathcal{R}} = \mathcal{R} \otimes_{\mathbb{C}} \mathbf{V}$  and set  $\mathbf{V}_{\mathcal{R}}^{r,s} = \mathcal{R} \otimes_{\mathbb{C}} \mathbf{V}^{r,s}$ .

It follows from [13, Thm. 5.1] that there is a natural epimorphism from the walled Brauer-Clifford superalgebra  $\text{BC}_{r,s}$  onto  $(\mathcal{R}/(q-1)\mathcal{R}) \otimes_{\mathcal{R}} \text{BC}_{r,s}(\mathcal{R}) \cong \text{BC}_{r,s}(\mathcal{R})/(q-1)\text{BC}_{r,s}(\mathcal{R})$  and hence a natural epimorphism

$$\pi : \text{BC}_{r,s} \twoheadrightarrow \text{BC}_{r,s}(\mathcal{R})/(q-1)\text{BC}_{r,s}(\mathcal{R}).$$

We want to argue that  $\pi$  is an isomorphism.

Let  $\rho_n^{r,s} : \text{BC}_{r,s} \rightarrow \text{End}_{\mathbb{C}}(\mathbf{V}^{r,s})^{\text{op}}$  be the representation given just before Theorem 1.4. The action of  $\text{BC}_{r,s}(q)$  on  $\mathbf{V}_q^{r,s}$  defined in Theorem 3.16 restricts to a representation  $\rho_{n,\mathcal{R}}^{r,s} : \text{BC}_{r,s}(\mathcal{R}) \rightarrow \text{End}_{\mathcal{R}}(\mathbf{V}_{\mathcal{R}}^{r,s})$ . Let  $\bar{\rho}_{n,\mathcal{R}}^{r,s}$  be the homomorphism

$$\text{BC}_{r,s}(\mathcal{R})/(q-1)\text{BC}_{r,s}(\mathcal{R}) \rightarrow \text{End}_{\mathcal{R}}(\mathbf{V}_{\mathcal{R}}^{r,s})/(q-1)\text{End}_{\mathcal{R}}(\mathbf{V}_{\mathcal{R}}^{r,s}).$$

Since  $\mathbf{V}_{\mathcal{R}}^{r,s}$  is a free  $\mathcal{R}$ -module, the algebra  $\text{End}_{\mathcal{R}}(\mathbf{V}_{\mathcal{R}}^{r,s})$  is also free; thus, it is possible to identify  $\text{End}_{\mathcal{R}}(\mathbf{V}_{\mathcal{R}}^{r,s})/(q-1)\text{End}_{\mathcal{R}}(\mathbf{V}_{\mathcal{R}}^{r,s})$  with  $\text{End}_{\mathbb{C}}(\mathbf{V}^{r,s})$ .

Let  $j$  be the anti-involution of  $\text{BC}_{r,s}$  which fixes each generator. Upon the previous identification, the composite  $\bar{\rho}_{n,\mathcal{R}}^{r,s} \circ \pi$  is equal to  $\rho_n^{r,s} \circ j$ , as can be checked from the action of the generators on the mixed tensor space given in Theorem 3.16. (Setting  $q = 1$  in the formula for  $S$  in (3.1) gives the identity map). When  $n \geq r + s$ , the map  $\rho_n^{r,s}$  is known to be injective by [13, Thm. 4.5]. Therefore, if  $n \geq r + s$ , then  $\bar{\rho}_{n,\mathcal{R}}^{r,s} \circ \pi$  must also be injective, hence so is  $\pi$ . In conclusion,  $\pi$  is an isomorphism.  $\square$

In the proof of Theorem 5.1 in [13], a vector space basis of the walled Brauer-Clifford superalgebra  $\text{BC}_{r,s}$  is constructed. In this section, we obtain a basis of  $\text{BC}_{r,s}(q)$  which specializes to the one in [13] when  $q \mapsto 1$  (in a suitable sense). Our basis is inspired by the basis of the quantum walled Brauer algebra  $\text{H}_{r,s}^n(q)$  constructed in Section 2 of [11] (see Corollary 3.26 below).

**Definition 3.22.** [11] A *monomial  $\mathbf{n}$  in normal form in the generators  $\mathbf{t}_1, \mathbf{t}_2, \dots, \mathbf{t}_{r-1}$*  is a product of the form  $\mathbf{n} = \mathbf{p}_1 \mathbf{p}_2 \cdots \mathbf{p}_{r-1}$ , where  $\mathbf{p}_i = \mathbf{t}_i^{-1} \mathbf{t}_{i-1}^{-1} \cdots \mathbf{t}_j^{-1}$  for some  $j$  with  $1 \leq j \leq i+1$ . (If  $j = i+1$ , then  $\mathbf{p}_i = 1$ .) A *monomial  $\mathbf{n}^*$  in normal form in the generators  $\mathbf{t}_1^*, \mathbf{t}_2^*, \dots, \mathbf{t}_{s-1}^*$*  is a product of the form  $\mathbf{n}^* = \mathbf{p}_1^* \mathbf{p}_2^* \cdots \mathbf{p}_{s-1}^*$ , where  $\mathbf{p}_i^* = \mathbf{t}_i^* \mathbf{t}_{i-1}^* \cdots \mathbf{t}_j^*$  for some  $j$  with  $1 \leq j \leq i+1$ . (If  $j = i+1$ , then  $\mathbf{p}_i^* = 1$ .)

**Definition 3.23.** Suppose that  $I = (i_1, \dots, i_a)$  with  $1 \leq i_1 < \dots < i_a \leq r$ ,  $J = (j_1, \dots, j_a)$  with  $0 \leq j_k \leq s-1$  for  $k = 1, \dots, a$ , and if  $k_1 \neq k_2$ , then  $j_{k_1} \neq j_{k_2}$ . Let  $\tilde{I} \subseteq I$  and  $\tilde{J} \subseteq \{1, 2, \dots, r+s\} \setminus J$ .

A monomial  $\mathbf{m}$  in normal form in  $\text{BC}_{r,s}(q)$  is one of the form

$$\mathbf{m} = \mathbf{c}_{\tilde{I}} \left( \prod_{k=1, \dots, a}^{\rightarrow} \mathbf{t}_{j_k}^* \mathbf{t}_{j_k-1}^* \cdots \mathbf{t}_1^* \mathbf{t}_{i_k}^{-1} \cdots \mathbf{t}_{r-2}^{-1} \mathbf{t}_{r-1}^{-1} \mathbf{e} \mathbf{t}_{r-1}^{-1} \mathbf{t}_{r-2}^{-1} \cdots \mathbf{t}_{i_k}^{-1} \mathbf{t}_1^* \cdots \mathbf{t}_{j_k-1}^* \mathbf{t}_{j_k}^* \right) \mathbf{c}_{\tilde{J}} \mathbf{n} \mathbf{n}^*,$$

where

- (1)  $\mathbf{n}$  is a monomial in normal form in the generators  $\mathbf{t}_1, \mathbf{t}_2, \dots, \mathbf{t}_{r-1}$ ;
- (2)  $\mathbf{n}^*$  is a monomial in normal form in the generators  $\mathbf{t}_1^*, \mathbf{t}_2^*, \dots, \mathbf{t}_{s-1}^*$ ;
- (3) the product is arranged from  $k = 1$  to  $k = a$  from left to right;
- (4)  $\mathbf{c}_{\tilde{I}}$  is the product of  $\mathbf{c}_i$  over  $i \in \tilde{I}$  in increasing order, and  $\mathbf{c}_{\tilde{J}}$  is defined similarly.
- (5) Moreover, it is required that if  $\mathbf{n} = \mathbf{p}_1 \mathbf{p}_2 \cdots \mathbf{p}_{r-1}$  and  $\mathbf{p}_i = \mathbf{t}_i^{-1} \mathbf{t}_{i-1}^{-1} \cdots \mathbf{t}_j^{-1}$ , then  $\tilde{\sigma}^{-1}(i_1) < \dots < \tilde{\sigma}^{-1}(i_a)$  where  $\tilde{\sigma} = \sigma_1 \sigma_2 \cdots \sigma_{r-1}$  and  $\sigma_i$  is the cycle  $(i+1 \ i \ i-1 \ \dots \ j+1 \ j)$ .

**Theorem 3.24.** The set  $\mathcal{B}$  of monomials  $\mathbf{m}$  in normal form is a basis of  $\text{BC}_{r,s}(q)$  over  $\mathbb{C}(q)$ .

*Proof.* As in **Step 1** of [13, Thm. 5.1], there are relations analogous to those labeled (1)-(8), except that words of strictly smaller length need to be added on one side of each equality. By using induction on the length of words, we can verify that the set  $\mathcal{B}$  spans  $\text{BC}_{r,s}(q)$  over  $\mathbb{C}(q)$ .

Since  $\text{BC}_{r,s}(\mathcal{R})$  is a finitely generated torsion-free  $\mathcal{R}$ -module, it is free over  $\mathcal{R}$ . Now by a standard argument in abstract algebra (cf. [9, Chap. 4, Thm. 5.11]), it follows that  $\mathcal{B}$  is linearly independent over  $\mathbb{C}(q)$ .  $\square$

**Corollary 3.25.** The dimension of  $\text{BC}_{r,s}(q)$  over  $\mathbb{C}(q)$  is  $(r+s)! 2^{r+s}$ .

*Proof.* This follows from Theorem 3.24, **Step 2** in the proof [13, Thm. 5.1], and [11, Lem. 1.7].  $\square$

**Corollary 3.26.** The subalgebra of  $\text{BC}_{r,s}(q)$  generated by  $\mathbf{t}_1, \dots, \mathbf{t}_{r-1}, \mathbf{c}_1, \dots, \mathbf{c}_r$  (resp. by  $\mathbf{t}_1^*, \dots, \mathbf{t}_{s-1}^*$  and  $\mathbf{c}_1^*, \dots, \mathbf{c}_s^*$ ) is isomorphic to the finite Hecke-Clifford superalgebra  $\text{HC}_r(q)$  (resp. to  $\text{HC}_s(q)$ ). The subalgebra generated by  $\mathbf{t}_1, \dots, \mathbf{t}_{r-1}, \mathbf{t}_1^*, \dots, \mathbf{t}_{s-1}^*$ , and  $\mathbf{e}$  is isomorphic to the quantum walled Brauer algebra  $\text{H}_{r,s}^0(q)$  in [11].

*Proof.* The first assertion follows from the fact that the set  $\{\mathbf{c}_{\tilde{I}} \mathbf{n}\}$ , as  $\tilde{I}$  ranges over the subsets of  $I = \{1, \dots, r\}$  and  $\mathbf{n}$  ranges over the monomials in normal form in  $\mathbf{t}_1, \mathbf{t}_2, \dots, \mathbf{t}_{r-1}$ , is a basis of  $\text{HC}_r(q)$  over  $\mathbb{C}(q)$ . For the second assertion, one can show that the set  $\mathcal{B}'$  consisting of the elements

$$\left( \prod_{k=1, \dots, a}^{\rightarrow} \mathbf{t}_{j_k}^* \mathbf{t}_{j_k-1}^* \cdots \mathbf{t}_1^* \mathbf{t}_{i_k}^{-1} \cdots \mathbf{t}_{r-2}^{-1} \mathbf{t}_{r-1}^{-1} \mathbf{e} \mathbf{t}_{r-1}^{-1} \mathbf{t}_{r-2}^{-1} \cdots \mathbf{t}_{i_k}^{-1} \mathbf{t}_1^* \cdots \mathbf{t}_{j_k-1}^* \mathbf{t}_{j_k}^* \right) \mathbf{n} \mathbf{n}^*,$$

where  $\mathbf{n}$  is a monomial in normal form in  $\mathbf{t}_1, \mathbf{t}_2, \dots, \mathbf{t}_{r-1}$  satisfying the condition (5) in Definition 3.23, and  $\mathbf{n}^*$  is a monomial in normal form in  $\mathbf{t}_1^*, \mathbf{t}_2^*, \dots, \mathbf{t}_{s-1}^*$ , spans  $\text{H}_{r,s}^0(q)$  over  $\mathbb{C}(q)$ . Since  $\dim_{\mathbb{C}(q)} \text{H}_{r,s}^0(q) = (r+s)!$  and  $|\mathcal{B}'| \leq (r+s)!$ , the set  $\mathcal{B}'$  is a basis of  $\text{H}_{r,s}^0(q)$  over  $\mathbb{C}(q)$ .  $\square$

We will frequently deduce properties of  $\text{BC}_{r,s}(q)$  from the corresponding properties of  $\text{BC}_{r,s}$  using the following well-known facts about specialization, which we prove here for convenience.

**Lemma 3.27.** Suppose  $\mathbf{R}$  is a Noetherian local integral domain whose maximal ideal is generated by a single element  $x \in \mathbf{R}$ . Let  $\psi : \mathbf{A} \rightarrow \mathbf{B}$  be a homomorphism of finitely generated  $\mathbf{R}$ -modules, and consider the corresponding induced homomorphism

$$\overline{\psi} : \mathbf{A}/x\mathbf{A} \rightarrow \mathbf{B}/x\mathbf{B}, \quad \overline{\psi}(a + x\mathbf{A}) = \psi(a) + x\mathbf{B}.$$

- (i) If  $\bar{\psi}$  is surjective, then  $\psi$  is surjective.
- (ii) If  $B$  is torsion free and  $\bar{\psi}$  is injective, then  $\psi$  is injective, and its cokernel is also torsion free.

*Proof.* (i) Let  $C$  be the cokernel of  $\psi$ . The right exact sequence  $A \xrightarrow{\psi} B \twoheadrightarrow C$  induces a right exact sequence  $A/xA \xrightarrow{\bar{\psi}} B/xB \twoheadrightarrow C/xC$ . By assumption,  $\bar{\psi}$  is surjective, so  $C/xC = 0$ . Thus  $C = 0$  by Nakayama's lemma.

(ii) Let  $K$  be the kernel of  $\psi$ . If  $k \in K$ , then  $\psi(k) + xA$  is in the kernel of  $\bar{\psi}$ , which is zero by assumption. Thus  $k \in xA$ , so  $k = xa$  for some  $a \in A$ . Thus  $x\psi(a) = \psi(k) = 0$ , so  $\psi(a) = 0$  since  $B$  is torsion free. Thus,  $a \in K$ , so  $K = xK$ . Again by Nakayama's lemma,  $K = 0$ , so  $\psi$  is injective. Finally, choose an  $(R/xR)$ -basis  $\mathcal{X}$  of  $A/xA$  and extend it to a basis  $\mathcal{X} \sqcup \mathcal{Y}$  of  $B/xB$  (here we are identifying  $\bar{\psi}(\mathcal{X})$  with  $\mathcal{X}$  by injectivity). By lifting these basis elements arbitrarily to  $A$  and  $B$ , we obtain a commutative diagram

$$\begin{array}{ccccc} R^{\mathcal{X}} & \xrightarrow{\quad} & R^{\mathcal{X}} \oplus R^{\mathcal{Y}} & & \\ \downarrow & & \downarrow & & \\ A & \xrightarrow{\quad} & B & \twoheadrightarrow & B/A \end{array}$$

where the top two modules are free over  $R$ , and the vertical maps induce isomorphisms of  $(R/xR)$ -vector spaces. By what we've shown so far,  $R^{\mathcal{X}} \rightarrow A$  is surjective and  $R^{\mathcal{X}} \oplus R^{\mathcal{Y}} \rightarrow B$  is an isomorphism. Therefore  $B/A \cong R^{\mathcal{Y}}$  is free, and in particular, is torsion free.  $\square$

We now show that  $BC_{r,s}(q)$  gives the centralizer of the action of  $\mathfrak{U}_q(\mathfrak{q}(n))$  on  $\mathbf{V}_q^{r,s}$ . We deduce this from the corresponding result in the classical case, which is proven in [13].

**Theorem 3.28.** Let  $\rho_{n,q}^{r,s} : BC_{r,s}(q) \rightarrow \text{End}_{\mathfrak{U}_q(\mathfrak{q}(n))}(\mathbf{V}_q^{r,s})$  be the representation of  $BC_{r,s}(q)$  coming from Theorem 3.16. Then  $\rho_{n,q}^{r,s}$  is surjective, and when  $n \geq r + s$ , it is an isomorphism.

*Proof.* Let  $\pi : BC_{r,s} \xrightarrow{\sim} (\mathcal{R}/(q-1)\mathcal{R}) \otimes_{\mathcal{R}} BC_{r,s}(\mathcal{R})$  be the isomorphism established in Proposition 3.21. Consider the following elements of  $\mathfrak{U}_q(\mathfrak{q}(n))$  for  $i, j \in I = \{\pm i \mid i = 1, \dots, n\}$  with  $i \leq j$ :

$$\tilde{u}_{ij} = (q-1)^{-1} \begin{cases} u_{ij} - 1 & \text{if } i = j, \\ u_{ij} & \text{if } i \neq j. \end{cases}$$

Let

$$\tilde{U} = \sum_{i \leq j} \tilde{u}_{ij} \otimes E_{ij} \in \mathfrak{U}_q(\mathfrak{q}(n)) \otimes_{\mathbb{C}} \text{End}_{\mathbb{C}}(\mathbf{V}).$$

By (3.1), the action of  $\tilde{u}_{ij}$  on  $\mathbf{V}_q$  lies in  $\text{End}_{\mathcal{R}}(\mathbf{V}_{\mathcal{R}}) \subseteq \text{End}_{\mathbb{C}(q)}(\mathbf{V}_q)$ . Moreover, under the surjection  $\text{End}_{\mathcal{R}}(\mathbf{V}_{\mathcal{R}}) \twoheadrightarrow \text{End}_{\mathbb{C}}(\mathbf{V})$  given by evaluation at  $q = 1$ , the action of  $\tilde{u}_{ij}$  maps to the action of the following element of  $\mathfrak{q}(n)$ :

$$u_{ij} = \begin{cases} (-1)^{|i|} E_{ii}^0 & \text{if } i = j, \\ (-1)^{|i|} 2E_{(-1)^{|j|}j, (-1)^{|i|}i}^{1-\delta_{|i|,|j|}} & \text{if } i < j. \end{cases}$$

Similarly, by (3.15), the action of  $\tilde{u}_{ij}$  on  $\mathbf{V}_q^*$  lies in  $\text{End}_{\mathcal{R}}(\mathbf{V}_{\mathcal{R}}^*)$  and maps to the action of  $u_{ij}$  on  $\mathbf{V}^*$  by evaluation at  $q = 1$ . Finally, since the coproduct on  $\mathfrak{U}_q(\mathfrak{q}(n))$  sends

$$\Delta(\tilde{U}) = \tilde{U}^{13} + \tilde{U}^{23} + (q-1)\tilde{U}^{13}\tilde{U}^{23},$$

the corresponding statements extend to the action of  $\tilde{u}_{ij}$  on  $\mathbf{V}_q^{r,s}$ .

Now let  $\text{End}_{\tilde{U}}(\mathbf{V}_{\mathcal{R}}^{r,s})$  denote the space of endomorphisms in  $\text{End}_{\mathcal{R}}(\mathbf{V}_{\mathcal{R}}^{r,s})$  which supercommute with the action of  $\tilde{u}_{ij}$  for all  $i \leq j$ . We will show that the  $\mathcal{R}$ -module homomorphism

$$\psi : \text{BC}_{r,s}(\mathcal{R}) \longrightarrow \text{End}_{\tilde{U}}(\mathbf{V}_{\mathcal{R}}^{r,s})$$

is surjective, and an isomorphism if  $n \geq r + s$ . Note that if  $X \in \text{End}_{\mathcal{R}}(\mathbf{V}_{\mathcal{R}}^{r,s})$  is such that  $(q-1)X$  supercommutes with  $u_{ij}$ , then  $X$  also supercommutes with  $u_{ij}$ . Therefore, the induced homomorphism

$$(\mathcal{R}/(q-1)\mathcal{R}) \otimes_{\mathcal{R}} \text{End}_{\tilde{U}}(\mathbf{V}_{\mathcal{R}}^{r,s}) \rightarrow \text{End}_{\mathbb{C}}(\mathbf{V}^{r,s})$$

is injective. Moreover since the elements  $\{u_{ij} \mid i \leq j\}$  generate  $\mathfrak{q}(n)$ , this map factors through  $\text{End}_{\mathfrak{q}(n)}(\mathbf{V}^{r,s})$ . We obtain the following diagram.

$$\begin{array}{ccccc} \text{BC}_{r,s} & \xrightarrow{\pi} & (\mathcal{R}/(q-1)\mathcal{R}) \otimes_{\mathcal{R}} \text{BC}_{r,s}(\mathcal{R}) & \xrightarrow{\text{id} \otimes \psi} & (\mathcal{R}/(q-1)\mathcal{R}) \otimes_{\mathcal{R}} \text{End}_{\tilde{U}}(\mathbf{V}_{\mathcal{R}}^{r,s}) \\ & \searrow & & \swarrow & \downarrow \\ & & \text{End}_{\mathfrak{q}(n)}(\mathbf{V}^{r,s}) & \xrightarrow{\quad} & \text{End}_{\mathbb{C}}(\mathbf{V}^{r,s}) \end{array}$$

Now Theorem 3.5 of [13] shows that the homomorphism  $\rho_n^{r,s} : \text{BC}_{r,s} \rightarrow \text{End}_{\mathfrak{q}(n)}(\mathbf{V}^{r,s})^{\text{op}}$  given by the  $\text{BC}_{r,s}$ -module action is surjective, and also injective for  $n \geq r + s$ . It follows that  $(\mathcal{R}/(q-1)\mathcal{R}) \otimes_{\mathcal{R}} \text{End}_{\tilde{U}}(\mathbf{V}_{\mathcal{R}}^{r,s}) \rightarrow \text{End}_{\mathfrak{q}(n)}(\mathbf{V}^{r,s})$  is an isomorphism for all  $n$ , so  $(\mathcal{R}/(q-1)\mathcal{R}) \otimes_{\mathcal{R}} \psi$  is surjective for all  $n$  and injective for  $n \geq r + s$ . Since  $\text{End}_{\tilde{U}}(\mathbf{V}_{\mathcal{R}}^{r,s})$  is torsion free, we conclude by Lemma 3.27 that

$$\text{BC}_{r,s}(\mathcal{R}) \rightarrow \text{End}_{\tilde{U}}(\mathbf{V}_{\mathcal{R}}^{r,s})$$

is surjective for all  $n$  and injective for  $n \geq r + s$ . Finally, since the  $\tilde{u}_{ij}$  generate  $\mathfrak{U}_q(\mathfrak{q}(n))$ , we have

$$\mathbb{C}(q) \otimes_{\mathcal{R}} \text{End}_{\tilde{U}}(\mathbf{V}_{\mathcal{R}}^{r,s}) = \text{End}_{\mathfrak{U}_q(\mathfrak{q}(n))}(\mathbf{V}_q^{r,s}).$$

Therefore, tensoring by  $\mathbb{C}(q)$ , we obtain the desired result.  $\square$

**Remark 3.29.** The following question is left open: Does  $\mathfrak{U}_q(\mathfrak{q}(n))$  surject onto  $\text{End}_{\text{BC}_{r,s}(q)}(\mathbf{V}_q^{r,s})$ ?

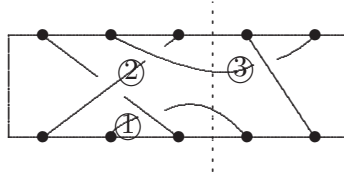
#### 4. THE $(r, s)$ -BEAD TANGLE ALGEBRAS $\text{BT}_{r,s}(q)$

In this section, we introduce a diagrammatic realization of the quantum walled Brauer-Clifford superalgebra  $\text{BC}_{r,s}(q)$  given in Definition 3.4.

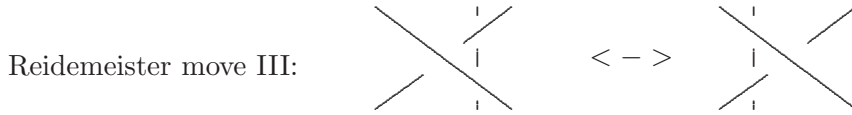
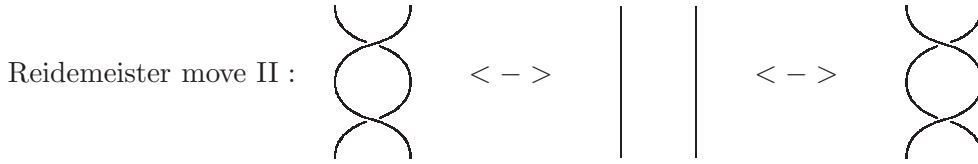
**Definition 4.1.** An  $(r, s)$ -bead tangle is a portion of a planar knot diagram in a rectangle  $R$  with the following conditions:

- (1) The top and bottom boundaries of  $R$  each have  $r + s$  vertices in some standard position.
- (2) There is a vertical wall that separates the first  $r$  vertices from the last  $s$  vertices on the top and bottom boundaries.
- (3) Each vertex must be connected to exactly one other vertex by an arc.
- (4) Each arc may (or may not) have finitely many numbered beads. The bead numbers in the tangle start with 1 and are distinct consecutive positive integers.
- (5) A *vertical arc* connects a vertex on the top boundary to a vertex on the bottom boundary of  $R$ , and it cannot cross the wall. A *horizontal arc* connects two vertices on the same boundary of  $R$ , and it must cross the wall.
- (6) An  $(r, s)$ -bead tangle may have finitely many loops.

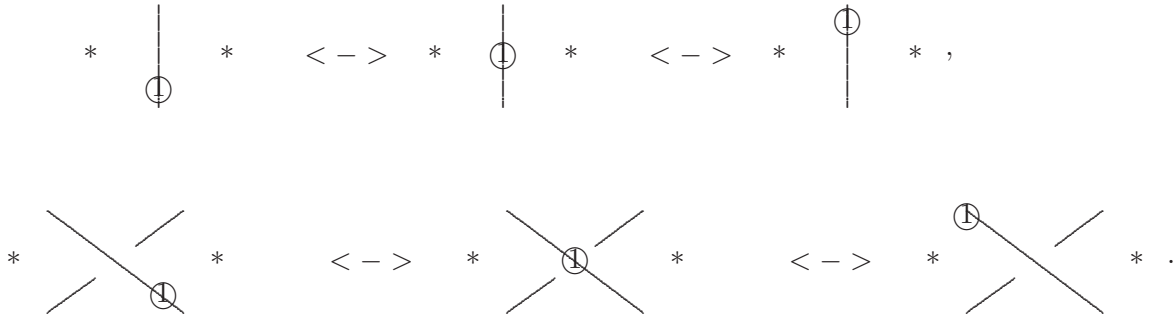
The following is an example of  $(3, 2)$ -bead tangle.



We want to stress that an  $(r, s)$ -bead tangle is in the plane, not in 3-dimensional space. We consider a bead as a point on the arc. Two  $(r, s)$ -bead tangles are *regularly isotopic* if they are related by a finite sequence of the Reidemeister moves II, III together with isotopies fixing the boundaries of  $R$ .



We observe that there are isotopies fixing the boundaries of the rectangles between the following tangles:



Therefore, moving a bead along a non-crossing arc or an over-crossing arc gives tangles that are regularly isotopic. We want to emphasize that the following are *not* regularly isotopic:



We identify an  $(r, s)$ -bead tangle with its regular isotopy class, and denote by  $\widetilde{\text{BT}}_{r,s}$  the set of  $(r, s)$ -bead tangles (up to regular isotopy). The  $(r, s)$ -bead tangle in which there are even (resp. odd) number of beads is regarded as *even* (resp. *odd*).

Now we define a multiplication on  $\widetilde{\text{BT}}_{r,s}$ . For  $(r,s)$ -bead tangles  $d_1, d_2$ , we place  $d_1$  under  $d_2$  and identify the top row of  $d_1$  with the bottom row of  $d_2$ . We add the largest bead number in  $d_1$  to each bead number in  $d_2$ , as we did for  $(r,s)$ -bead diagrams, and then concatenate the tangles. For example, if

$$d_1 = \begin{array}{|c|} \hline \text{Diagram 1} \\ \hline \end{array}, \quad d_2 = \begin{array}{|c|} \hline \text{Diagram 2} \\ \hline \end{array},$$

then,

$$d_1 d_2 = \begin{array}{|c|} \hline \text{Diagram 3} \\ \hline \end{array} = \begin{array}{|c|} \hline \text{Diagram 4} \\ \hline \end{array}$$

We observe  $\widetilde{\text{BT}}_{r,s}$  is closed under this product, and it is  $\mathbb{Z}_2$ -graded. The shape of the arcs are the same in  $(d_1 d_2) d_3$  and  $d_1 (d_2 d_3)$ . Moreover, the locations of the beads and the bead numbers are also the same in  $(d_1 d_2) d_3$  and  $d_1 (d_2 d_3)$ . It follows that the multiplication on  $\widetilde{\text{BT}}_{r,s}$  is associative. Hence  $\widetilde{\text{BT}}_{r,s}$  is a monoid with identity element

$$\begin{array}{|c|} \hline \text{Identity Element Diagram} \\ \hline \end{array}.$$

For  $1 \leq i \leq r-1$ ,  $1 \leq j \leq s-1$ ,  $1 \leq k \leq r$ ,  $1 \leq l \leq s$ , we define the following  $(r,s)$ -bead tangles:

$$\begin{aligned} \sigma_i &:= \begin{array}{|c|} \hline \text{Diagram } \sigma_i \\ \hline \end{array}, & \sigma_j^* &:= \begin{array}{|c|} \hline \text{Diagram } \sigma_j^* \\ \hline \end{array}, \\ h &:= \begin{array}{|c|} \hline \text{Diagram } h \\ \hline \end{array}, \\ c_k &:= \begin{array}{|c|} \hline \text{Diagram } c_k \\ \hline \end{array}, & c_l^* &:= \begin{array}{|c|} \hline \text{Diagram } c_l^* \\ \hline \end{array}. \end{aligned}$$

From Reidemeister move II, we obtain the following elements in  $\widetilde{\text{BT}}_{r,s}$ :



$$\sigma_i^{-1} = \begin{array}{c} \text{---} i \quad i+1 \text{---} \\ \text{---} \end{array} \quad , \quad (\sigma_j^*)^{-1} = \begin{array}{c} \text{---} j \quad j+1 \text{---} \\ \text{---} \end{array} .$$

Now we consider the submonoid  $\widetilde{\text{BT}}'_{r,s}$  of  $\widetilde{\text{BT}}_{r,s}$  generated by  $\sigma_i^{\pm 1}, (\sigma_j^*)^{\pm 1}, h, c_k, c_l^*$ . We denote by  $\text{BT}'_{r,s}(q)$  the monoid algebra of  $\widetilde{\text{BT}}'_{r,s}$  over  $\mathbb{C}(q)$  and define the algebra  $\text{BT}_{r,s}(q)$  to be the quotient of  $\text{BT}'_{r,s}(q)$  by the following relations (for allowable  $i, j$ ):

$$(4.2) \quad \begin{aligned} \sigma_i^{-1} &= \sigma_i - (q - q^{-1}), & (\sigma_j^*)^{-1} &= \sigma_j^* - (q - q^{-1}), \\ h\sigma_{r-1}h &= h, \quad h^2 = 0, & h\sigma_1^*h &= h, \quad hc_rh = 0, \\ c_i^2 &= -1, \quad c_ic_j = -c_jc_i \quad (i \neq j), \quad c_ic_j^* = -c_j^*c_i, & (c_i^*)^2 &= 1, \quad c_i^*c_j^* = -c_i^*c_j^* \quad (i \neq j). \end{aligned}$$

For simplicity, we identify the coset of a diagram in  $\text{BT}_{r,s}(q)$  with the diagram itself. Note that we get extra terms when a bead moves along an under-crossing arc. That is, we have

$$\begin{aligned} \text{---} \text{---} & \quad < - > \quad \text{---} \text{---} + (q - q^{-1}) \begin{array}{c} | \\ \text{---} \end{array} \quad | \\ \text{---} \text{---} & \quad < - > \quad \text{---} \text{---} + (q - q^{-1}) \begin{array}{c} | \\ \text{---} \end{array} \quad | \\ \text{---} \text{---} & \quad < - > \quad \text{---} \text{---} + (q - q^{-1}) \left( \begin{array}{c} | \\ \text{---} \end{array} \quad \begin{array}{c} | \\ \text{---} \end{array} - \begin{array}{c} | \\ \text{---} \end{array} \quad \begin{array}{c} | \\ \text{---} \end{array} \right), \end{aligned}$$

which is equivalent to  $c_i\sigma_i = \sigma_i c_{i+1} + (q - q^{-1})(c_i - c_{i+1})$ . Similarly, we obtain  $c_{i+1}\sigma_i^{-1} = \sigma_i^{-1}c_i + (q - q^{-1})(c_i - c_{i+1})$ . We call  $\text{BT}_{r,s}(q)$  the  $(r, s)$ -bead tangle algebra or simply the *bead tangle algebra*.

In [10], Kauffman introduced the algebra of tangles and showed that it is isomorphic to the *Birman-Murakami-Wenzl algebra*. To show that  $\text{BT}_{r,s}(q)$  is isomorphic to  $\text{BC}_{r,s}(q)$ , we will follow the outline of the argument given in [10, Thm. 4.4].

Let  $F'_{r,s}$  be the monoid generated by  $t_1^{\pm 1}, t_2^{\pm 1}, \dots, t_{r-1}^{\pm 1}, (t_1^*)^{\pm 1}, (t_2^*)^{\pm 1}, \dots, (t_{s-1}^*)^{\pm 1}, e, c_1, c_2, \dots, c_r$  and  $c_1^*, c_2^*, \dots, c_s^*$  with the following defining relations (for  $i, j$  in the allowable range):

$$\begin{aligned} (4.3) \quad t_i t_i^{-1} &= t_i^{-1} t_i = 1, & t_j^* (t_j^*)^{-1} &= (t_j^*)^{-1} t_j^* = 1, \\ (4.4) \quad t_i t_{i+1} t_i &= t_{i+1} t_i t_{i+1}, & t_i^* t_{i+1}^* t_i^* &= t_{i+1}^* t_i^* t_{i+1}^*, \\ (4.5) \quad t_i t_j &= t_j t_i \quad (|i - j| > 1), & t_i^* t_j^* &= t_j^* t_i^* \quad (|i - j| > 1), \\ (4.6) \quad t_i t_j^* &= t_j^* t_i, & & \\ (4.7) \quad e t_j &= t_j e \quad (j \neq r - 1), & e t_j^* &= t_j^* e \quad (j \neq 1), \\ (4.8) \quad e t_{r-1}^{-1} t_1^* e &= e t_{r-1}^{-1} t_1^* e t_1^* t_{r-1}^{-1} = t_{r-1}^{-1} t_1^* e t_{r-1}^{-1} t_1^* e, & & \end{aligned}$$

$$(4.9) \quad \begin{aligned} et_{r-1}(t_1^*)^{-1}e &= et_{r-1}(t_1^*)^{-1}e(t_1^*)^{-1}t_{r-1} \\ &= t_{r-1}(t_1^*)^{-1}et_{r-1}(t_1^*)^{-1}e, \end{aligned}$$

$$(4.10) \quad et_{r-1}^{-1}t_1^*e = et_{r-1}(t_1^*)^{-1}e,$$

$$(4.11) \quad t_i c_i = c_{i+1} t_i,$$

$$t_i^* c_i^* = c_{i+1}^* t_i^*,$$

$$(4.12) \quad t_i c_j = c_j t_i \quad (j \neq i, i+1),$$

$$t_i^* c_j^* = c_j^* t_i^* \quad (j \neq i, i+1),$$

$$(4.13) \quad t_i c_j^* = c_j^* t_i,$$

$$t_i^* c_j = c_j t_i^*,$$

$$(4.14) \quad c_r e = c_1^* e,$$

$$ec_r = ec_1^*,$$

$$(4.15) \quad c_j e = ec_j \quad (j \neq r),$$

$$c_j^* e = ec_j^* \quad (j \neq 1).$$

We define a monoid homomorphism  $\varphi_{r,s} : F'_{r,s} \rightarrow \mathbf{BT}'_{r,s}$  by

$$\varphi_{r,s}(t_i^{\pm 1}) = \sigma_i^{\pm 1}, \quad \varphi_{r,s}((t_j^*)^{\pm 1}) = (\sigma_j^*)^{\pm 1}, \quad \varphi_{r,s}(e) = h, \quad \varphi_{r,s}(c_i) = c_i, \quad \text{and} \quad \varphi_{r,s}(c_j^*) = c_j^*.$$

By direct computations, one can check that  $\sigma_i^{\pm 1}, (\sigma_j^*)^{\pm 1}, h, c_k, c_l^*$  satisfy the corresponding defining relations (4.3) - (4.15) in  $\mathbf{BT}'_{r,s}$ . Moreover,  $\varphi_{r,s}$  is surjective.

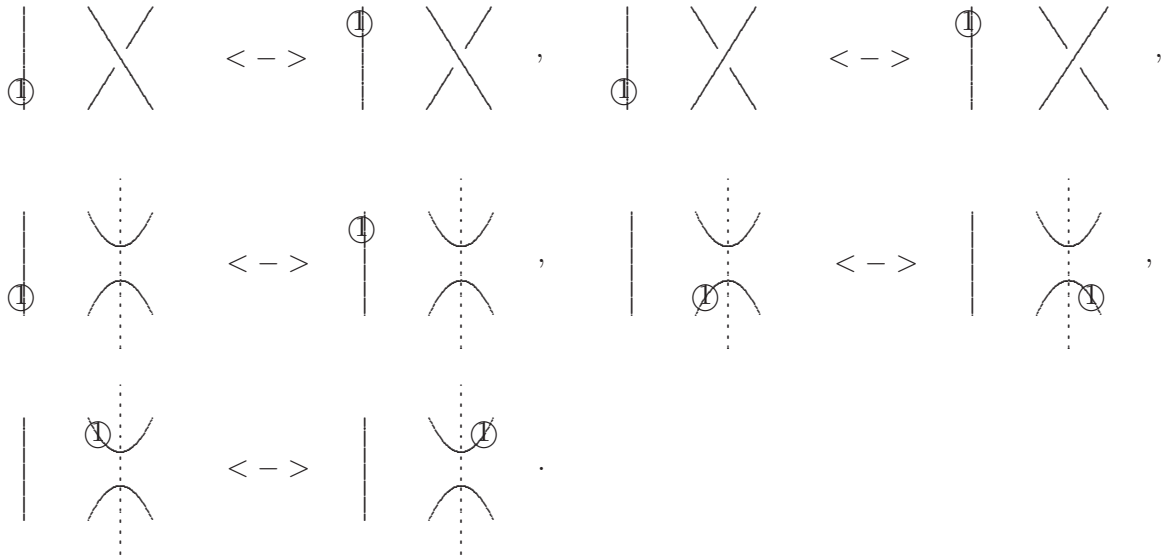
**Theorem 4.16.** The monoid  $F'_{r,s}$  is isomorphic to  $\mathbf{BT}'_{r,s}$  as monoids.

*Proof.* It suffices to show that  $\varphi_{r,s}$  is injective. Assume that  $\varphi_{r,s}(d) = \varphi_{r,s}(d') \in \mathbf{BT}'_{r,s}$ . This means that  $\varphi_{r,s}(d')$  can be obtained from  $\varphi_{r,s}(d)$  by a finite sequence of Reidemeister moves II, III and vice versa.

If  $\varphi_{r,s}(d')$  can be obtained from  $\varphi_{r,s}(d)$  without moving beads, then modifying the proof of [10, Thm. 4.4], we can show that  $d'$  can be obtained from  $d$  using relations (4.3) - (4.10).

We consider the various cases in which we need to move the beads.

**Case 1:** A bead moves along a non-crossing arc. We have the following five cases.



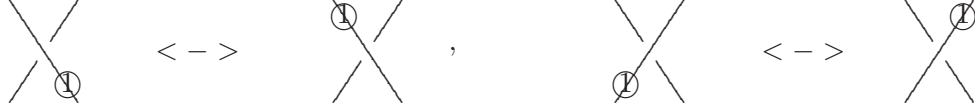
Observe that the above are equivalent to the following relations for  $j \neq i, i+1$ ,  $l \neq r$ ,  $m \neq 1$ , and allowable values of  $k$ :

$$c_j \sigma_i = \sigma_i c_j, \quad c_j \sigma_k^* = \sigma_k^* c_j, \quad c_j \sigma_i^{-1} = \sigma_i^{-1} c_j, \quad c_j (\sigma_k^*)^{-1} = (\sigma_k^*)^{-1} c_j,$$

$$\begin{aligned}
c_j^* \sigma_i^* &= \sigma_i^* c_j^*, & c_j^* \sigma_k &= \sigma_k c_j^*, & c_j^* (\sigma_i^*)^{-1} &= (\sigma_i^*)^{-1} c_j^*, & c_j^* \sigma_k^{-1} &= \sigma_k^{-1} c_j^*, \\
c_l h &= h c_l, & c_m^* h &= h c_m^*, & c_r h &= c_1^* h, & h c_r &= h c_1^*.
\end{aligned}$$

Hence  $d$  and  $d'$  are related in  $F'_{r,s}$ .

**Case 2:** A bead moves along an over-crossing arc. In the following two cases,

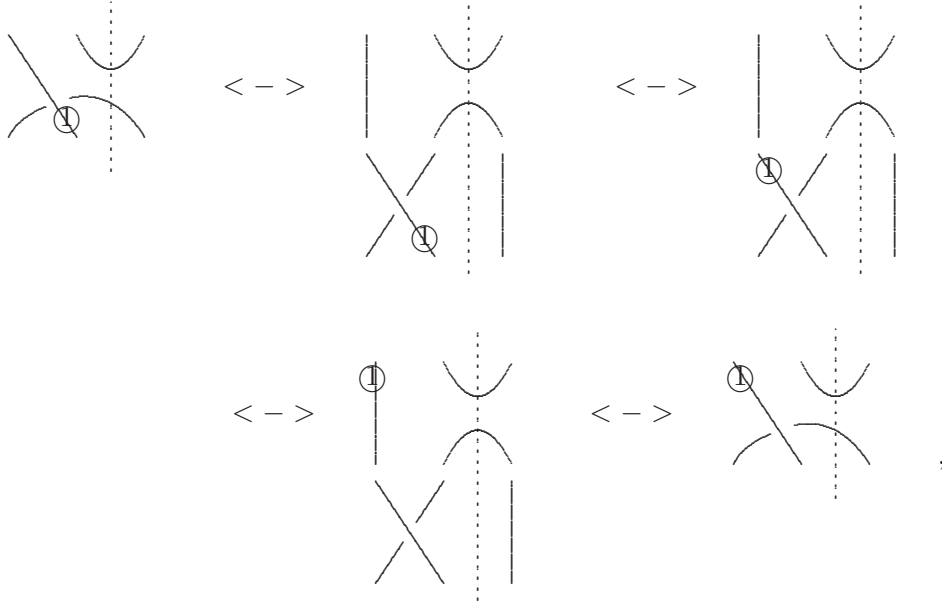


the corresponding relations are equivalent to

$$c_{i+1} \sigma_i = \sigma_i c_i, \quad c_{i+1}^* \sigma_i^* = \sigma_i^* c_i^*, \quad c_i \sigma_i^{-1} = \sigma_i^{-1} c_{i+1}, \quad c_i^* (\sigma_i^*)^{-1} = (\sigma_i^*)^{-1} c_{i+1}^*,$$

which implies that  $d$  and  $d'$  are related.

The remaining cases appear as a mixture of **Case 1** and **Case 2**. For instance, a crossing of a horizontal and a vertical arc is a combination of the following moves.



which can be written as  $c_2 \sigma_1 h = \sigma_1 c_1 h = \sigma_1 h c_1$ .

In conclusion, when  $\varphi_{r,s}(d')$  is obtained from  $\varphi_{r,s}(d)$  by moving beads,  $d$  can be transformed to  $d'$  by the corresponding relations in (4.11) - (4.15).  $\square$

Let  $F'_{r,s}(q) = \mathbb{C}(q)F'_{r,s}$  be the associated monoid algebra, and let  $R$  be the two-sided ideal of  $F'_{r,s}(q)$  corresponding to the following relations:

$$\begin{aligned}
(4.17) \quad t_i^{-1} &= t_i - (q - q^{-1}), & (t_j^*)^{-1} &= t_j^* - (q - q^{-1}), \\
et_{r-1}e &= e, \quad e^2 = 0, & et_1^*e &= e, \quad ec_re = 0, \\
c_i^2 &= -1, \quad c_i c_j = -c_j c_i \quad (i \neq j), \quad c_i c_j^* = -c_j^* c_i. & (c_i^*)^2 &= 1, \quad c_i^* c_j^* = -c_j^* c_i^* \quad (i \neq j).
\end{aligned}$$

We consider  $(t_i)^{\pm 1}, (t_j^*)^{\pm 1}, h$  as the *even* generators and  $c_i, c_j^*$  as the *odd* generators.

We denote by  $F_{r,s}(q)$  the quotient superalgebra  $F'_{r,s}(q)/R$ . For ease of notation, we also use  $t_i, t_j^*, h, c_k$  and  $c_l^*$  for the generators of  $F_{r,s}(q)$ . We note that the relations in (4.2) correspond to the relations in (4.17) via the map  $\varphi_{r,s}$ . By the definitions of  $F_{r,s}(q)$  and  $\text{BT}_{r,s}(q)$  and Theorem 4.16, we obtain the following corollary.

**Corollary 4.18.** The superalgebra  $F_{r,s}(q)$  is isomorphic to  $\text{BT}_{r,s}(q)$  as associative superalgebras.

One can check that relations (4.3) - (4.15) and (4.17) include the corresponding relations (3.5) if we map  $t_i, t_j, e, c_k$  and  $c_l^*$  to  $t_i, t_j, h, c_k$  and  $c_l^*$ , respectively. Using the relations in (4.17) and Remark 3.8, we obtain that the relations corresponding to (4.8) - (4.10) are also satisfied in  $\text{BC}_{r,s}(q)$ . It follows that  $F_{r,s}(q)$  is isomorphic to  $\text{BC}_{r,s}(q)$  as associative superalgebras. Therefore, we obtain the following main result of this section.

**Theorem 4.19.** The quantum walled Brauer-Clifford superalgebra  $\text{BC}_{r,s}(q)$  is isomorphic to the  $(r, s)$ -bead tangle algebra  $\text{BT}_{r,s}(q)$  as associative superalgebras.

**Corollary 4.20.** The dimension of  $\text{BT}_{r,s}(q)$  over  $\mathbb{C}(q)$  is  $(r + s)!2^{r+s}$ .

**Remark 4.21.** Since  $\text{BC}_{r,s}$  is the classical limit of  $\text{BC}_{r,s}(q)$ , by Theorem 2.9 and Theorem 4.19, we conclude that  $\text{BD}_{r,s}$  is the classical limit of  $\text{BT}_{r,s}(q)$ .

## 5. THE $q$ -SCHUR SUPERALGEBRA OF TYPE Q AND ITS DUAL

There are two equivalent ways to define the  $q$ -Schur algebra  $S_q(n; \ell)$  associated to  $\mathfrak{U}_q(\mathfrak{gl}(n))$ : either as the image of  $\mathfrak{U}_q(\mathfrak{gl}(n))$  in  $\text{End}_{\mathbb{C}(q)}((\mathbb{C}(q)^n)^{\otimes \ell})$  or as  $\text{End}_{H_\ell(q)}((\mathbb{C}(q)^n)^{\otimes \ell})$ , where  $H_\ell(q)$  is the Hecke algebra (the subalgebra of  $\text{HC}_\ell(q)$  generated by the  $t_i$ ,  $i = 1, \dots, \ell - 1$ ). Analogous definitions can be considered in our quantum super context, but we are not able to prove that they are equivalent. Therefore, in order to develop a viable theory, we have settled on the following definition.

**Definition 5.1.** The  $q$ -Schur superalgebra of type Q, denoted  $S_q(n; r, s)$ , is  $\text{End}_{\text{BC}_{r,s}(q)}(\mathbf{V}_q^{r,s})$ .

Even when  $s = 0$ , this superalgebra had not been studied until the recent paper [6]. In this case, it follows from [18, Thm. 5.3] that the next result holds, but we don't know if it is true for arbitrary  $s \geq 1$ .

**Proposition 5.2.**  $S_q(n; r, 0)$  is equal to the image of  $\mathfrak{U}_q(\mathfrak{q}(n))$  in  $\text{End}_{\mathbb{C}(q)}(\mathbf{V}_q^{r,0})$ .

There is a third point of view on  $q$ -Schur algebras adopted for instance in [5], which is as duals of certain homogeneous subspaces of quantum matrix algebras. Super analogues of quantum  $\text{GL}_n$  were first introduced in [17] and more general superalgebras were studied in [8], where bases were constructed using quantum minors and indexed by standard bitableaux. In this section, we obtain similar results for a quantum matrix superalgebra of type Q.

Let

$$\delta_{i < j} = \begin{cases} 1 & \text{if } i < j, \\ 0 & \text{otherwise} \end{cases}$$

and  $\delta_{i \pm j} = \delta_{ij} + \delta_{i, -j}$ . Also recall that  $\xi = q - q^{-1}$ .

**Definition 5.3.** We denote by  $A_q(n)$  the associative unital algebra over  $\mathbb{C}(q)$  generated by  $x_{ab}$  and  $\bar{x}_{ab}$  for  $1 \leq a, b \leq n$ , subject to the following relations for any  $1 \leq a, b, c, d \leq n$  with  $a \leq c$ :

$$q^{\delta_{ac}} x_{ab} x_{cd} = q^{\delta_{bd}} x_{cd} x_{ab} + \xi \delta_{b < d} x_{cb} x_{ad} + \xi \bar{x}_{cb} \bar{x}_{ad},$$

$$\begin{aligned}
q^{\delta_{ac}} x_{ab} \bar{x}_{cd} &= q^{-\delta_{bd}} \bar{x}_{cd} x_{ab} - \xi \delta_{d < b} \bar{x}_{cb} x_{ad}, \\
q^{\delta_{ac}} \bar{x}_{ab} x_{cd} &= q^{\delta_{bd}} x_{cd} \bar{x}_{ab} + \xi \bar{x}_{cb} x_{ad} + \xi \delta_{b < d} x_{cb} \bar{x}_{ad}, \\
q^{\delta_{ac}} \bar{x}_{ab} \bar{x}_{cd} &= -q^{-\delta_{bd}} \bar{x}_{cd} \bar{x}_{ab} + \xi \delta_{d < b} \bar{x}_{cb} \bar{x}_{ad}.
\end{aligned}$$

We define a  $\mathbb{Z}$ -grading on  $A_q(n)$  by declaring each generator to have degree 1. We call  $A_q(n)$  the *quantum matrix superalgebra of type Q*.

**Remark 5.4.** The quotient of  $A_q(n)$  by the two-sided ideal generated by the odd elements is isomorphic to the quantum matrix algebra as presented for instance in Section 1.3 of [2] with  $v = q^{-1}$ .

The algebra  $A_q(n)$  can be viewed as a  $q$ -deformation of the algebra of polynomial functions on the space  $\mathcal{M}_n(\mathbb{Q})$  of  $(2n \times 2n)$ -matrices of type Q inside  $\mathcal{M}_{n|n}(\mathbb{C})$ . A  $q$ -deformation of the algebra of polynomial functions on  $\mathcal{M}_{n|n}(\mathbb{C})$  was first given in [17].

**Lemma 5.5.** The algebra  $A_q(n)$  is isomorphic to the unital associative algebra over  $\mathbb{C}(q)$  generated by elements  $x_{ij}$  with  $i, j \in \mathbb{I} = \{\pm 1, \dots, \pm n\}$ , which satisfy the relations  $x_{ij} = x_{-i, -j}$  and

$$(5.6) \quad S^{23} X^{12} X^{13} = X^{13} X^{12} S^{23}$$

where  $S^{23}$  is the same matrix used in Definition 3.2.

*Proof.* This follows from relations (5.10) and (5.11) below and from the proof of Theorem 5.8.  $\square$

**Corollary 5.7.** The algebra  $A_q(n)$  is a bialgebra with coproduct  $\Delta$  given by

$$\Delta(x_{ij}) = \sum_{\substack{k=-n \\ k \neq 0}}^n (-1)^{(|i|+|k|)(|j|+|k|)} x_{ik} \otimes x_{kj}.$$

**Theorem 5.8.** Let  $A_q(n, r)$  denote the degree  $r$  component of  $A_q(n)$ . There is a vector space isomorphism  $A_q(n, r) \xrightarrow{\sim} \text{End}_{\text{HC}_r(q)}(\mathbf{V}_q^{\otimes r})^*$ . Explicitly, let  $\{E_{ij}^\vee\}$  denote the basis of  $\text{End}_{\mathbb{C}(q)}(\mathbf{V}_q)^*$  dual to the natural basis  $\text{End}_{\mathbb{C}(q)}(\mathbf{V}_q)$ . Define a map  $A_q(n, 1) \rightarrow \text{End}_{\mathbb{C}(q)}(\mathbf{V}_q)^*$  by  $x_{ab} \rightarrow E_{ab}^\vee$ ,  $\bar{x}_{ab} \rightarrow E_{a, -b}^\vee$ . This extends to the map  $A_q(n, r) \rightarrow \text{End}_{\text{HC}_r(q)}(\mathbf{V}_q^{\otimes r})^*$  via the (super) identification

$$(\text{End}_{\mathbb{C}(q)}(\mathbf{V}_q)^*)^{\otimes r} \cong \text{End}_{\mathbb{C}(q)}(\mathbf{V}_q^{\otimes r})^*$$

*Proof.* Let  $F_q(n)$  denote the free algebra generated by  $\varepsilon_{ij}$  for  $i, j \in \mathbb{I} = \{\pm 1, \pm 2, \dots, \pm n\}$ , and let  $F_q(n, r)$  denote the degree  $r$  component of  $F_q(n)$  where each generator has degree 1. Sending  $\varepsilon_{ij}$  to  $E_{ij}^\vee$ , we obtain a vector space isomorphism  $F_q(n, r) \cong \text{End}_{\mathbb{C}(q)}(\mathbf{V}_q^{\otimes r})^*$ . An element of  $\text{End}_{\mathbb{C}(q)}(\mathbf{V}_q^{\otimes r})^*$  will lie in the subspace  $\text{End}_{\text{HC}_r(q)}(\mathbf{V}_q^{\otimes r})^*$  if its coefficients satisfy certain relations. We can obtain  $\text{End}_{\text{HC}_r(q)}(\mathbf{V}_q^{\otimes r})^*$  by quotienting  $F_q(n, r)$  by the same relations.

The generator  $c_k$  of  $\text{HC}_r(q)$  acts on the  $k$ th tensor factor via the map  $\Phi$  (3.17), and the generator  $t_k$  of  $\text{HC}_r(q)$  acts on the  $k$ th and  $(k+1)$ st factors via the map  $PS$  (3.18). We compute the supercommutator of each of these maps with an arbitrary endomorphism.

$$\begin{aligned}
\left[ \Phi, \sum_{ij} a_{ij} E_{ij} \right] &= \sum_{ij} (-1)^{|i|} (a_{-i, -j} - a_{ij}) E_{i, -j}. \\
\left[ PS, \sum_{ijkl} a_{ijkl} E_{ij} \otimes E_{kl} \right] &= \sum_{ijkl} a_{ijkl} \left[ (-1)^{|i|+(|i|+|k|)(|i|+|j|)} q^{\delta_{i \pm k}} (-1)^{|k|} E_{kj} \otimes E_{il} \right.
\end{aligned}$$

$$\begin{aligned}
& + \xi \delta_{i < k} E_{ij} \otimes E_{kl} + \xi \delta_{-i < k} (-1)^{|i|+|k|+|i|+|j|} E_{-i,j} \otimes E_{-k,l} \\
& - (-1)^{|l|+(|j|+|l|)(|k|+|l|)} q^{\delta_{j \pm l}} (-1)^{|j|} E_{il} \otimes E_{kj} \\
& - \xi \delta_{j < l} E_{ij} \otimes E_{kl} - \xi \delta_{j < -l} (-1)^{|j|+|l|+|k|+|l|} E_{i,-j} \otimes E_{k,-l} \Big] \\
(5.9) \quad & = \sum_{ijkl} \Big[ (-1)^{|i|(|j|+|k|)+|j||k|} q^{\delta_{i \pm k}} (-1)^{|k|} a_{ijkl} \\
& + \xi \delta_{k < i} a_{kjil} - \xi \delta_{k < -i} (-1)^{|i|+|j|} a_{-k,j,-i,l} \\
& - (-1)^{|j|(|l|+|i|)+|l||i|} q^{\delta_{j \pm l}} (-1)^{|l|} a_{kl ij} \\
& - \xi \delta_{j < l} a_{kjil} + \xi \delta_{-j < l} (-1)^{|i|+|j|} a_{k,-j,i,-l} \Big] E_{kj} \otimes E_{il}.
\end{aligned}$$

Therefore,  $\text{End}_{\text{HC}_r(q)}(\mathbf{V}_q^{\otimes r})^*$  is the degree  $r$  component of  $\mathbf{A}_q(n)' = \mathbf{F}_q(n)/\langle R(ij), R(ijkl) \rangle$  by [4, Lemma 2.3], where we have factored out by the ideal generated by the elements

$$(5.10) \quad R(ij) = \varepsilon_{-i,-j} - \varepsilon_{ij},$$

$$\begin{aligned}
(5.11) \quad R(ijkl) &= q^{\delta_{i \pm k}} (-1)^{|k|} (-1)^{(|i|+|j|)|l|} \varepsilon_{ij} \varepsilon_{kl} \\
&\quad - q^{\delta_{j \pm l}} (-1)^{|l|} (-1)^{(|i|+|j|)|k|} \varepsilon_{kl} \varepsilon_{ij} \\
&\quad + \xi (-1)^{|k||i|+|k||l|+|j||l|} (\delta_{k < i} - \delta_{j < l}) \varepsilon_{kj} \varepsilon_{il} \\
&\quad + \xi (-1)^{|k||i|+|k||l|+|j||l|+|k|+|l|} (\delta_{k < -i} \varepsilon_{-k,j} \varepsilon_{-i,l} - \delta_{-j < l} \varepsilon_{k,-j} \varepsilon_{i,-l}).
\end{aligned}$$

The element  $R(ijkl)$  was obtained from the right-hand side of (5.9) by multiplying it by  $(-1)^{|i||j|}$  and by replacing each  $a_{ijkl}$  by  $(-1)^{(|i|+|j|)(|k|+|l|)} \varepsilon_{ij} \varepsilon_{kl}$ . It remains to show that  $\mathbf{A}_q(n)' \cong \mathbf{A}_q(n)$ . Let  $\mathbf{F}_q(n)'$  be the free algebra generated by  $x_{ab}$  and  $\bar{x}_{ab}$  for  $1 \leq a, b \leq n$ . For convenience, we define elements  $x_{ij}$  and  $\bar{x}_{ij}$  in  $\mathbf{F}_q(n)'$  for all  $i, j \in \mathbf{I}$  such that

$$x_{-i,j} = \bar{x}_{ij} = x_{i,-j}.$$

Clearly there is an isomorphism  $\mathbf{F}_q(n)/\langle R(ij) \rangle \cong \mathbf{F}_q(n)'$  sending  $\varepsilon_{ij}$  to  $x_{ij}$ . Let  $R(ijkl)'$  be the image of  $R(ijkl)$  in  $\mathbf{F}_q(n)'$ . Note that

$$\begin{aligned}
R(-i,-j,k,l)' &= (-1)^{|l|(|i|+|j|)} q^{\delta_{i \pm k}} (-1)^{|k|} x_{ij} x_{kl} - (-1)^{|k|(|i|+|j|)} q^{\delta_{j \pm l}} (-1)^{|l|} x_{kl} x_{ij} \\
&\quad + (-1)^{|k||i|+|k||l|+|j||l|+|k|+|l|} \xi (\delta_{k < -i} - \delta_{-j < l}) \bar{x}_{kj} \bar{x}_{il} \\
&\quad + (-1)^{|k||i|+|k||l|+|j||l|} \xi (\delta_{k < i} - \delta_{j < l}) x_{kj} x_{il} \\
&= R(ijkl)'.
\end{aligned}$$

Similarly, using  $q^{\delta_{i \pm k}} (-1)^{|k|} - q^{-\delta_{i \pm k}} (-1)^{|k|} = \delta_{i \pm k} (-1)^{|k|} \xi$ , we have

$$\begin{aligned}
(-1)^{|i|+|j|} R(i,j,-k,-l)' &= (-1)^{|l|(|i|+|j|)} q^{-\delta_{i \pm k}} (-1)^{|k|} x_{ij} x_{kl} - (-1)^{|k|(|i|+|j|)} q^{-\delta_{j \pm l}} (-1)^{|l|} x_{kl} x_{ij} \\
&\quad - (-1)^{|k||i|+|k||l|+|j||l|+|k|+|l|} \xi (1 - \delta_{i < -k} - \delta_{i,-k} - 1 + \delta_{-l < j} + \delta_{-l,j}) \bar{x}_{kj} \bar{x}_{il} \\
&\quad - (-1)^{|k||i|+|k||l|+|j||l|} \xi (1 - \delta_{k < i} - \delta_{ik} - 1 + \delta_{j < l} + \delta_{jl}) x_{kj} x_{il} \\
&= (-1)^{|l|(|i|+|j|)} q^{\delta_{i \pm k}} (-1)^{|k|} x_{ij} x_{kl} - \delta_{i \pm k} (-1)^{|k|+|l|(|i|+|j|)} \xi x_{ij} x_{kl} \\
&\quad - (-1)^{|k|(|i|+|j|)} q^{\delta_{j \pm l}} (-1)^{|l|} x_{kl} x_{ij} + \delta_{j \pm l} (-1)^{|k|(|i|+|j|)+|l|} \xi x_{kl} x_{ij} \\
&\quad + (-1)^{|k||i|+|k||l|+|j||l|+|k|+|l|} \xi (\delta_{i < -k} - \delta_{-l < j} + \delta_{i,-k} - \delta_{-l,j}) \bar{x}_{kj} \bar{x}_{il} \\
&\quad + (-1)^{|k||i|+|k||l|+|j||l|} \xi (\delta_{k < i} - \delta_{j < l} + \delta_{i,k} - \delta_{j,l}) x_{kj} x_{il} \\
&= R(ijkl)'.
\end{aligned}$$

Observe also that  $q^{\delta_{i\pm k}((-1)^{|k|+(-1)^{|i|}})} - 1 = \delta_{ik}(-1)^{|k|}q^{(-1)^{|k|}}\xi$ . Therefore

$$\begin{aligned}
& (-1)^{|j||l|}q^{\delta_{i\pm k}(-1)^{|i|}}R(ijkl)' + (-1)^{|i||k|}q^{\delta_{j\pm l}(-1)^{|l|}}R(klij)' \\
&= (-1)^{|i||l|}\left(q^{\delta_{i\pm k}((-1)^{|k|+(-1)^{|i|}})} - q^{\delta_{j\pm l}((-1)^{|j|+(-1)^{|l|}})}\right)x_{ij}x_{kl} \\
&\quad + (-1)^{|k|(|i|+|l|)}q^{\delta_{i\pm k}(-1)^{|i|}}\xi(\delta_{k<i} - \delta_{j<l})x_{kj}x_{il} \\
&\quad + (-1)^{|k||-i|+|l||-k|}q^{\delta_{i\pm k}(-1)^{|i|}}\xi(\delta_{k<-i} - \delta_{-j<l})\bar{x}_{kj}\bar{x}_{il} \\
&\quad + (-1)^{(|i|+|l|)|j|}q^{\delta_{j\pm l}(-1)^{|l|}}\xi(\delta_{i<k} - \delta_{l<j})x_{il}x_{kj} \\
&\quad + (-1)^{|i||-j|+|j||-l|}q^{\delta_{j\pm l}(-1)^{|l|}}\xi(\delta_{i<-k} - \delta_{-l<j})\bar{x}_{il}\bar{x}_{kj} \\
&= (-1)^{|k|(|i|+|l|)}q^{\delta_{i\pm k}(-1)^{|i|}}\xi(\delta_{k<i} + \delta_{ik} - \delta_{j<l})x_{kj}x_{il} \\
&\quad + (-1)^{(|i|+|l|)|j|}q^{\delta_{j\pm l}(-1)^{|l|}}\xi(\delta_{i<k} - \delta_{l<j} - \delta_{lj})x_{il}x_{kj} \\
&\quad + \xi(\delta_{i<-k} - \delta_{-j<l}) \times \\
&\quad \left[(-1)^{|k||-i|+|l||-k|}q^{\delta_{i\pm k}(-1)^{|i|}}\bar{x}_{kj}\bar{x}_{il} + (-1)^{|i||-j|+|j||-l|}q^{\delta_{j\pm l}(-1)^{|l|}}\bar{x}_{il}\bar{x}_{kj}\right] \\
&= \xi(1 - \delta_{i<k} - \delta_{j<l}) \times \\
&\quad \left[(-1)^{|k|(|i|+|l|)}q^{\delta_{i\pm k}(-1)^{|i|}}x_{kj}x_{il} - (-1)^{(|i|+|l|)|j|}q^{\delta_{j\pm l}(-1)^{|l|}}x_{il}x_{kj}\right] \\
&\quad + \xi(\delta_{i<-k} - \delta_{-j<l}) \times \\
&\quad \left[(-1)^{|k||-i|+|l||-k|}q^{\delta_{i\pm k}(-1)^{|i|}}\bar{x}_{kj}\bar{x}_{il} + (-1)^{|i||-j|+|j||-l|}q^{\delta_{j\pm l}(-1)^{|l|}}\bar{x}_{il}\bar{x}_{kj}\right].
\end{aligned}$$

On the other hand, note that if  $\delta_1, \delta_2 \in \{0, 1\}$  then  $(\delta_1 - \delta_2)(1 - \delta_1 - \delta_2) = \delta_1 - \delta_2 - \delta_1^2 + \delta_2^2 = 0$ , so

$$\begin{aligned}
(1 - \delta_{i<k} - \delta_{j<l})R(kjil)' &= (1 - \delta_{i<k} - \delta_{j<l}) \times \\
&\quad \left[(-1)^{|l|(|k|+|j|)}q^{\delta_{i\pm k}(-1)^{|i|}}x_{kj}x_{il} - (-1)^{|i|(|k|+|j|)}q^{\delta_{j\pm l}(-1)^{|l|}}x_{il}x_{kj}\right] \\
&\quad + (-1)^{|i||k|+|i||l|+|j||l|+|i|+|l|}\xi(1 - \delta_{i<k} - \delta_{j<l})(\delta_{i<-k} - \delta_{-j<l})\bar{x}_{ij}\bar{x}_{kl}.
\end{aligned}$$

Similarly

$$\begin{aligned}
R(k, -j, i, -l)' &= (-1)^{|-l|(|k|+|-j|)}q^{\delta_{i\pm k}(-1)^{|i|}}\bar{x}_{kj}\bar{x}_{il} - (-1)^{|i|(|k|+|-j|)}q^{\delta_{j\pm l}(-1)^{|l|}}\bar{x}_{il}\bar{x}_{kj} \\
&\quad + (-1)^{|i||k|+|i||l|+|j||l|+|i|+|j|+|l|}\xi(1 - \delta_{i<k} - \delta_{j<l})\bar{x}_{ij}\bar{x}_{kl} \\
&\quad - (-1)^{|i||k|+|i||l|+|j||l|+|j|}\xi(1 - \delta_{i<-k} - \delta_{-j<l})x_{ij}x_{kl},
\end{aligned}$$

so

$$\begin{aligned}
-(-1)^{|j||l|+|j|+|i||k|}(\delta_{i<-k} - \delta_{-j<l})R(k, -j, i, -l)' &= (\delta_{i<-k} - \delta_{-j<l}) \times \\
&\quad \left[(-1)^{|k||-i|+|l||-k|}q^{\delta_{i\pm k}(-1)^{|i|}}\bar{x}_{kj}\bar{x}_{il} + (-1)^{|i||-j|+|j||-l|}q^{\delta_{j\pm l}(-1)^{|l|}}\bar{x}_{il}\bar{x}_{kj}\right] \\
&\quad + (-1)^{|-i||-l|}\xi(1 - \delta_{i<k} - \delta_{j<l})(\delta_{i<-k} - \delta_{-j<l})\bar{x}_{ij}\bar{x}_{kl}.
\end{aligned}$$

Thus,

$$\begin{aligned}
& (-1)^{|j||l|}q^{\delta_{i\pm k}(-1)^{|i|}}R(ijkl)' + (-1)^{|i||k|}q^{\delta_{j\pm l}(-1)^{|l|}}R(klij)' \\
&= \xi(-1)^{|k||i|+|l||j|}(1 - \delta_{i<k} - \delta_{j<l})R(kjil)' \\
&\quad - \xi(-1)^{|j|+|j||l|+|k||i|}(\delta_{i<-k} - \delta_{-j<l})R(k, -j, i, -l)'.
\end{aligned}$$

These dependencies amongst the  $R(ijkl)'$  imply that

$$\text{span}_{\mathbb{C}(q)}\{R(ijkl)' \mid i, j, k, l \in \mathbf{I}\} = \text{span}_{\mathbb{C}(q)}\{R(ijkl)' \mid i, j, k, l \in \mathbf{I} \text{ with } 0 < i \leq k\}.$$



Note that if  $0 < i \leq k$ , then  $R(ijkl)'$  simplifies to

$$\begin{aligned} R(ijkl)' &= q^{\delta_{i \pm k}} (-1)^{|j||l|} x_{ij} x_{kl} - q^{\delta_{j \pm l}} (-1)^{|l|} x_{kl} x_{ij} \\ &\quad - (-1)^{|j||l|} \xi \delta_{j < l} x_{kj} x_{il} - (-1)^{|j||l|+|l|} \xi \delta_{-j < l} \bar{x}_{kj} \bar{x}_{il}. \end{aligned}$$

By considering the four possibilities for  $(|j|, |l|)$ , we obtain the four relations in the definition of  $A_q(n)$ . Thus

$$A_q(n)' = F_q(n)' / \langle R(ijkl)' \mid i, j, k, l \in \mathbb{I} \text{ with } 0 < i \leq k \rangle = A_q(n).$$

□

It is also possible to prove an analogue of Theorem 5.8 when  $s \neq 0$  using the coalgebra  $A_q(n; r, s)$  that we define immediately below. Set  $x_{ab}^* = x_{ab}$  and  $\bar{x}_{ab}^* = \sqrt{-1} \bar{x}_{ab}$ . The relations in the following definition are super analogues of those in Lemma 4.1 of [4].

**Definition 5.12.** Abbreviate  $x_{ab} \otimes 1$  and  $1 \otimes x_{ab}^*$  in  $A_q(n) \otimes_{\mathbb{C}(q)} A_{q^{-1}}(n)$  by  $x_{ab}$  and  $x_{ab}^*$ , respectively. Then  $\tilde{A}_q(n)$  is defined to be the quotient of  $A_q(n) \otimes_{\mathbb{C}(q)} A_{q^{-1}}(n)$  by the two-sided ideal generated by the following:

$$(5.13) \quad \sum_{e=1}^n q^{2e} (x_{eb} x_{ed}^* - \bar{x}_{eb} \bar{x}_{ed}^*) \text{ for } b \neq d, \quad \sum_{e=1}^n q^{2e} (x_{eb} \bar{x}_{ed}^* - \bar{x}_{eb} x_{ed}^*),$$

$$(5.14) \quad \sum_{e=1}^n (x_{ae} x_{ce}^* + \bar{x}_{ae} \bar{x}_{ce}^*) \text{ for } a \neq c, \quad \sum_{e=1}^n (x_{ae} \bar{x}_{ce}^* - \bar{x}_{ae} x_{ce}^*),$$

$$(5.15) \quad \sum_{e=1}^n q^{2e-2b} (x_{eb} x_{eb}^* - \bar{x}_{eb} \bar{x}_{eb}^*) - \sum_{e=1}^n (x_{ae} x_{ae}^* + \bar{x}_{ae} \bar{x}_{ae}^*).$$

**Definition 5.16.**  $A_q(n; r, s)$  is defined as the subspace of  $\tilde{A}_q(n)$  spanned by monomials in the generators of bidegree  $(r, s)$ ; that is, of degree  $r$  in the generators  $x_{ab}, \bar{x}_{ab}$  and of degree  $s$  in the generators  $x_{ab}^*, \bar{x}_{ab}^*$ .

**Theorem 5.17.** There is a vector space isomorphism  $A_q(n; r, s) \xrightarrow{\sim} \text{End}_{\text{BC}_{r,s}(q)}(\mathbf{V}_q^{r,s})^*$ .

*Proof.* Most of the necessary computations are already contained in the proof of Theorem 5.8. Recall that  $\cap \cup = q^{-(2n+1)} \sum_{i,j \in \mathbb{I}} (-1)^{|i||j|} q^{2j(1-2|j|)} E_{ij} \otimes E_{ij}$ . We only need to explain where the new relations (5.13)-(5.15) in Definition 5.12 come from, and for this we have to compute the following commutator:

$$\begin{aligned} \left[ \cap \cup, \sum_{ijkl} a_{ijkl} E_{ij} \otimes E_{kl} \right] &= q^{-(2n+1)} \sum_{ijl} \left( \sum_p a_{pjpl} q^{2p(1-2|p|)} (-1)^{|p|+|p||j|} \right) (-1)^{|i||j|} E_{ij} \otimes E_{il} \\ &\quad - q^{-(2n+1)} \sum_{ikl} \left( \sum_p a_{ipkp} q^{2l(1-2|l|)} (-1)^{|p|+|k||p|} \right) (-1)^{|k||l|} E_{il} \otimes E_{kl} \end{aligned}$$

This leads to the relation

$$\delta_{ik} \sum_p (-1)^{|p||l|+|j||l|} q^{2p(1-2|p|)} x_{pj} x_{pl}^* = \delta_{jl} \sum_p (-1)^{|i||k|+|i||p|} q^{2l(1-2|l|)} x_{ip} x_{kp}^*.$$

The relations (5.13)-(5.15) can be deduced by considering the cases  $i \neq k$  and  $j = l$ ;  $i = k$  and  $j \neq l$ ;  $i = k$  and  $j = l$ ; and also the various possibilities for the signs of  $i, j, k, l$ . Note that

$x_{ab}^* = x_{-a,-b}^*$  and  $x_{a,-b}^* = -x_{-a,b}^* = \bar{x}_{ab}^*$  for  $1 \leq a, b \leq n$ , because of the relation  $[\Phi^T, \sum_{ij} a_{ij} E_{ij}] = \sum_{ij} (a_{ij} - (-1)^{|i|+|j|} a_{-i,-j}) E_{-i,j}$ .  $\square$

**Remark 5.18.** The algebra  $A_q(n; r, s)$  could possibly be used to prove the open problem of showing the surjectivity of the map  $\mathcal{U}_q(\mathfrak{q}(n)) \rightarrow \text{End}_{\text{BC}_{r,s}(q)}(\mathbf{V}_q^{r,s})$  (see Remark 3.29). The following line of reasoning was applied in [4] to establish a similar surjectivity result for  $\mathcal{U}_q(\mathfrak{gl}(n))$  over a quite general base ring. First, it might be possible to obtain a homomorphism  $\tau : S_q(n; r', 0) \rightarrow S_q(n; r, s)$  for some  $r'$  (possibly  $r' = r + s$ ) via some embedding of the mixed tensor space into  $\mathbb{C}(n|n)^{\otimes r'}$ . The surjectivity of the map  $\mathcal{U}_q(\mathfrak{q}(n)) \rightarrow \text{End}_{\text{BC}_{r,s}(q)}(\mathbf{V}_q^{r,s})$  then would follow from the surjectivity of  $\tau$ , which is equivalent to the injectivity of  $\tau^* : S_q(n; r, s)^* \rightarrow S_q(n; r', 0)^*$ . As suggested by [4], the injectivity of  $\tau^*$  could perhaps be shown by constructing bases of  $A_q(n; r, s)$  and  $A_q(n; r', 0)$  using super analogues of bideterminants. For the quantum general linear supergroup, this was accomplished in [8].

**Remark 5.19.** The algebra  $A_q(n)$  is a bialgebra, so it can be enlarged to a Hopf algebra, the so-called *Hopf envelope* of  $A_q(n)$ . This is explained in [17] in the context of the quantum general linear supergroup attached to  $\mathfrak{gl}(m|n)$ . Moreover, it is proved in [17] that the Hopf envelope of  $A_q(m|n)$ , the quantized algebra of functions on the space of super matrices of size  $(m|n) \times (m|n)$ , is isomorphic to the localization of  $A_q(m|n)$  with respect to the quantum Berezinian, a super analogue of the quantum determinant. This localization is the quantized algebra of functions  $\mathbb{C}_q[\text{GL}_{m|n}]$ . This raises the following question: is the Hopf envelope of  $A_q(n)$  isomorphic to the localization of  $A_q(n)$  with respect to an appropriate super version of the quantum determinant? Such a localization could be thought of as a quantized algebra of functions for the supergroup of type  $Q_n$ .

## REFERENCES

- [1] G. Benkart, M. Chakrabarti, T. Halverson, R. Leduc, C. Lee and J. Stroome, *Tensor product representations of general linear groups and their connections with Brauer algebras*, J. Algebra **166** (1994), 529–567.
- [2] J. Brundan, R. Dipper, A. Kleshchev, *Quantum Linear Groups and Representations of  $GL_n(\mathbf{F}_q)$* , Mem. Amer. Math. Soc. **149** (2001), no. 706.
- [3] R. Dipper, S. Doty, F. Stoll, *The quantized walled Brauer algebra and mixed tensor space*, Algebr. Represent. Theory **17** (2014), no. 2, 675–701.
- [4] R. Dipper, S. Doty, F. Stoll, *Quantized mixed tensor space and Schur-Weyl duality*, Algebra Number Theory **7** (2013), 1121–1146.
- [5] S. Donkin, *The  $q$ -Schur Algebra*, London Mathematical Society Lecture Note Series, **253**, Cambridge University Press, Cambridge, 1998.
- [6] J. Du, J. Wan, *Presenting queer Schur superalgebras*, Int. Math. Res. Not. IMRN 2015, no. 8, 2210–2272.
- [7] T. Halverson, *Characters of the centralizer algebras of mixed tensor representations of  $GL(r, \mathbb{C})$  and the quantum group  $\mathcal{U}_q(\mathfrak{gl}(r, \mathbb{C}))$* , Pacific J. Math. **174** (1996), no. 2, 359 – 410.
- [8] R.Q. Huang, J.J. Zhang, *Standard basis theorem for quantum linear groups*, Adv. Math. **102** (1993), no. 2, 202–229.
- [9] T.W. Hungerford, *Algebra*, Graduate Texts in Mathematics, Vol. 73, Springer-Verlag, New York, 1974.
- [10] L.H. Kauffman, *An invariant of regular isotopy*, Trans. Amer. Math. Soc. **318** (1990), 417 – 471.
- [11] M. Kosuda, J. Murakami, *Centralizer algebras of the mixed tensor representations of quantum group  $\mathcal{U}_q(\mathfrak{gl}(n, \mathbb{C}))$* , Osaka J. Math. **30** (1993), no. 3, 475–507.
- [12] K. Koike, *On the decomposition of tensor products of the representations of classical groups: by means of universal characters*, Adv. Math. **74** (1989), 57–86.
- [13] J.H. Jung, S.-J. Kang, *Mixed Schur-Weyl-Sergeev duality for queer Lie superalgebras*, J. Algebra **399** (2014), 516–545.
- [14] R. Leduc, *A Two-parameter Version of the Centralizer Algebra of the Mixed Tensor Representation of the General Linear Group and Quantum General Linear Group*, PhD thesis, University of Wisconsin-Madison, 1994.
- [15] C. Lee Shader, D. Moon, *Mixed tensor representations and rational representations for the general linear Lie superalgebras*, Comm. Algebra **30** (2002), no. 2, 839–857.
- [16] C. Lee Shader, D. Moon, *Mixed tensor representations of quantum superalgebra  $\mathcal{U}_q(\mathfrak{gl}(m, n))$* , Comm. Algebra **35** (2007), no. 3, 781–806.

- [17] Y.I. Manin, *Topics in noncommutative geometry*, M. B. Porter Lectures. Princeton University Press, Princeton, NJ, 1991. viii+164 pp.
- [18] G. Olshanski, *Quantized universal enveloping superalgebra of type  $Q$  and a super-extension of the Hecke algebra*, Lett. Math. Phys. **24** (1992), no. 2, 93–102.
- [19] A.N. Sergeev, *Tensor algebra of the identity representation as a module over the Lie superalgebras  $Gl(n, m)$  and  $Q(n)$* , (Russian) Mat. Sb. (N.S.) **123(165)** (1984), no. 3, 422–430.
- [20] V. Turaev, *Operator invariants of tangles and  $R$ -matrices*, (Russian) Izv. Akad. Nauk SSSR Ser. Mat. 53 (1989), no. 5, 1073–1107, 1135; translation in Math. USSR-Izv. 35 (1990), no. 2, 411–444.

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF WISCONSIN-MADISON, MADISON, WI 53706-1325, USA

*E-mail address:* `benkart@math.wisc.edu`

DEPARTMENT OF MATHEMATICAL AND STATISTICAL SCIENCES, UNIVERSITY OF ALBERTA, CAB 632, EDMONTON, AB T6G 2G1, CANADA

*E-mail address:* `nguy@ualberta.ca`

DEPARTMENT OF MATHEMATICAL SCIENCES AND RESEARCH INSTITUTE OF MATHEMATICS, SEOUL NATIONAL UNIVERSITY, SEOUL 151-747, REPUBLIC OF KOREA

*E-mail address:* `jung.ji.hye@hotmail.com`, `sjkang@snu.ac.kr`

SAN FRANCISCO, CA, USA

*E-mail address:* `stewbasic@gmail.com`